LOVELY PROFESSIONAL UNIVERSITY

OPERATIONS RESEARCH

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SYLLABUS

Operations Research

Objectives: To introduce the students to the basic concepts of Operations Techniques and their applications to Business decision problems

Operations Research

Sr. No.	Description
1.	**Operations Research:** Meaning, significance and scope; History of OR, applications of OR; OR Models.
2.	**Linear Programming Problems (LPP):** introduction, problem formulation, graphical solutions.
3.	LPP-simplex method, Big M method, unconstrained variables, sensitivity analysis, Duality.
4.	**Transportation Problems:** Introduction, transportation model, north west corner method (NWCM), row and column minima (LCET), VAM, optimality test-stepping stone, and Modi method.
5.	**Assignment Problems:** Introduction, Hungarian method. Typical assignment problems like optimal assignment of crews and travelling salesman problem.
6.	**Game Theory:**Introduction, two persons zero sum games, pure strategies, saddle point, mixed strategies, Dominance Method.
7.	**Sequencing Problems:** Introduction, processing jobs through two machines, three machines. Replacement theory
8.	**Queuing Theory:** concept, waiting line process, single server queuing model (M/M/1) only.
9.	**CPM and PERT:** introduction, time estimates, slack, float, finding critical paths, problem solving.
10.	**Inventory Control:** only deterministic model,Decision making under certainty, under risk and under uncertainty. Expected value, EVPI, decision tree analysis.

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Unit 1: Introduction to Operations Research

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1.10 Further Readings

Objectives

After studying this unit, you will be able to:

- Understand the meaning of Operations research
- Know about the history of operations research
- Discuss the scope and application of operations research
- Explain the various types of models used in operations research

Introduction

Operations Research has gained wider acclaim in the modern complex business world. For every complex problem of an industry today, well defined Operations Research tools are the solace. Decision making in today's social and business environment has become a complex task. Well structured problems are routinely optimized at the operational level of organizations, and increased attention is now focused on broader tactical and strategic issues. To effectively address this problems and provide leadership in the advancing global age, decision makers cannot afforded to make decisions by simply applying their personal experiences, guesswork's or intuitions, because consequences of wrong decisions are serious and costly. Hence, an understanding of the applicability of the quantitative methods to decision making is of fundamental importance to decision makers. Operations research is one such tool that helps in comparison of every possible alternative (course of action) to know the potential outcomes, permits examination of the sensitivity of the solution to changes or errors.

This unit introduces the basics of operation research to help students to get an overview of operations research and thereby help in better understanding of the subsequent units.

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1.1 Meaning of Operations Research

OR can be defined as:

OR is the application of the methods of science to complex problem arising in the direction and management of large system of men, machines, materials and money in industry, business, government of defense. The distinctive approach is to develop a scientific model of the system, incorporating measurements of factors such as chance & risk, with which to predict and compare the outcome of alternative decisions strategies and controls. The purpose is to help management determine its policies and actions scientifically.

— Journal of OR Society of UK

"OR is an experimental and applied science devoted to observing, understanding and predicting the behaviour of purposeful man-machine systems; and OR workers are actively engaged in applying this knowledge to practical problems in business, government and society"

— OR Society of America

This definition is not widely accepted and often criticized as it emphasizes on complex problems and large systems indicating that OR can be used in highly technical and sophisticated environment of large organizations.

In operations research, problems are broken down into basic components and then solved in defined steps by mathematical analysis. Analytical methods used in OR include mathematical logic, simulation, network analysis, queuing theory, and game theory.

1.2 Significance of Operations Research

The basic question that comes to the mind is what is the importance of OR. Is knowledge of OR techniques really required? Can't we do away with the OR techniques in our decision making process?

The answer to all these questions is quite simple and lies in the understanding of the significance of OR. We have to know about different OR techniques because of the following reasons.

- The problem may be complex for which normal guess does not work. Or techniques help in arriving at a feasible solution to the problem that too with full analysis and research.
- The problem is very important and no lethargy on part of the decision-maker can be entertained. In such a situation also, OR comes to rescue. OR methodologies are so detailed and meticulous that even a little scope for leaving a point unattended is unaffordable.
- The problem is new and no precedence exists to facilitate logical and intelligent decision making of the problem. In this situation, OR comes to the rescue of analysts and the reason for this is that OR has a large pool of techniques that aid to solve even new and unthought-of problems.
- The problem is repetitive and a model may be developed to enhance faster and better decisions. Again OR helps like nothing else to aid the analysts in such a situation by offering either a readymade model, or a readymade process to develop such a model and pacify the decision making process.

Self Assessment

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Multiple Choice Questions:

1. Operations research approach is

   (a) Multidisciplinary (b) Scientific

   (c) Intuitive (d) All of the above

2. Operations research analysts do not

   (a) Predict future operations (b) Build more than one model

   (c) Collect relevant data (d) Recommend decision and accept

3. For analyzing a problem, decision makers should normally study its

   (a) Quantitative aspect (b) Qualitative aspects

   (c) Both (a) and (b) (d) Neither (a) nor (b)

1.3 Scope of Operations Research

Operations Research addresses a wide variety of issues in transportation, inventory planning, production planning, communication operations, computer operations, financial assets, risk management, revenue management, and many other fields where improving business productivity is paramount.

As OR has made (over the years) significant contributions in virtually all industries, in almost all managerial and decision-making functions, and at most organizational levels, the list of OR applications is prodigious. Successful OR applications can be found in a broad array of industries dealing with challenges such as planning, routing, scheduling, forecasting, process analysis and decision analysis.

OR assists decision-makers in almost any management function. To illustrate, OR supports the key decision making process, allows to solve urgent problems, can be utilized to design improved multi-step operations (processes), setup policies, supports the planning and forecasting steps, and measures actual results.

1. ***Manufacturing :*** OR's success in contemporary business pervades manufacturing and service operations, logistics, distribution, transportation, and telecommunication.

   Operations research is used to for various activities which include scheduling, routing, workflow improvements, elimination of bottlenecks, inventory control, business process re-engineering, site selection, or facility and general operational planning.

   OR helps in developing software for material flow analysis and design for flexible manufacturing facilities, using pattern recognition and graph theory algorithms. Further, approaches for the design of re-configurable manufacturing systems and progressive automation of discrete manufacturing systems are under development. Additional OR projects focus on the industrial deployment of computer-based methods for assembly line balancing, business process reengineering, capacity planning, pull scheduling, and setup reduction.

2. ***Revenue Management:*** The application of OR in revenue management entails

   - first to accurately forecasting the demand, and
   - secondly to adjust the price structure over time to more profitably allocate fixed capacity.

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3. ***Supply Chain Management:*** In the area of Supply Chain Management, OR helps in taking decisions that include the who, what, when, and where abstractions from purchasing and transporting raw materials and parts, through manufacturing actual products and goods, and finally distributing and delivering the items to the customers.

   The primary objective here is to reduce overall cost while processing customer orders more efficiently than before. The power of utilizing OR methods allows examining a rather complex and convoluted chain in a comprehensive manner, and to search among a vast number of combinations for the resource optimization and allocation strategy that seem most effective, and hence beneficial to the operation.

4. ***Retailing:*** In supermarkets, data from store loyalty card schemes is analyzed by OR groups to advice on merchandising policies and profitability improvement.

   OR methods are also used to decide when and where new store developments should be made.

5. ***Financial Services:*** In financial markets, OR practitioners address issues such as portfolio and risk management and planning and analysis of customer service. They are widely employed in Credit Risk Management—a vital area for lenders needing to ensure that they find the optimum balance of risk and revenue.

   OR techniques are also applied in cash flow analysis and capital budgeting.

6. ***Marketing Management:*** OR helps marketing manager in making the apt selection of product mix. It helps them in making optimum sales resource allocation and assignments.

7. ***Human Resource Management:*** OR techniques are being applied widely in the functional area of Human Resource Management by helping the human resource managers in activities like manpower planning, resource allocation, staffing and scheduling of training programs.

8. ***General Management:*** OR helps in designing Decision Support System and management of information systems, organizational design and control, software process management and Knowledge Management.

9. ***Production systems:*** The area of operations research that concentrates on real-world operational problems is known as production systems. Production systems problems may arise in settings that include, but are not limited to, manufacturing, telecommunications, health-care delivery, facility location and layout, and staffing.

The area of production systems presents special challenges for operations researchers. Production problems are operations research problems, hence solving them requires a solid foundation in operations research fundamentals. Additionally, the solution of production systems problems frequently draws on expertise in more than one of the primary areas of operations research, implying that the successful production researcher cannot be one-dimensional.

Production systems problems cannot be solved without an in-depth understanding of the real problem, since invoking assumptions that simplify the mathematical structure of the problem may lead to an elegant solution for the wrong problem. Common sense and practical insight are common attributes of successful production planners.

At the current time, the field of OR is extremely dynamic and ever evolving. To name a few of the contemporary (primary) research projects, current work in, primarily through the integration of the philosophies of the Theory of Constraints and Lean Manufacturing.

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Notes Many real life OR models require long and complex mathematical calculations. Thus, computer software packages that are used to do these calculations rapidly and effectively have become part of OR approach to problem solving.

Some of the well known computer software used for OR problems are QSB+(Quantitaive system for Business Plus), QSOM (Quantitative system for operations Management), Value STORM, TORA and LINDO (Linear Interactive Discrete Optimization).

Self Assessment

Fill in the blanks:

4. OR techniques are also applied in cash flow analysis and …….
5. The area of operations research that concentrates on real-world operational problems is known as ……
6. The area of production systems presents special challenges for ……

1.4 History of Operations Research

A brief history of OR how it originated along with some of the problems faced and subsequently solved by the experts on OR is presented below which has been adopted from the notes of J E Beasley, Imperial College.

1936

Early in 1936 the British Air Ministry established Bawdsey Research Station, on the east coast, near Felixstowe, Suffolk, as the centre where all pre-war radar experiments for both the Air Force and the Army would be carried out. Experimental radar equipment was brought up to a high state of reliability and ranges of over 100 miles on aircraft were obtained.

It was also in 1936 that Royal Air Force (RAF) Fighter Command, charged specifically with the air defence of Britain, was first created. It lacked however any effective fighter aircraft and no radar data was yet fed into its very elementary warning and control system.

It had become clear that radar would create a whole new series of problems in fighter direction and control, so in late 1936 some experiments started at Biggin Hill in Kent into the effective use of such data. This early work, attempting to integrate radar data with ground based observer data for fighter interception, was the start of OR.

1939

In the summer of 1939, Britain held what was to be its last pre-war air defence exercise. It involved some 33,000 men, 1,300 aircraft, 110 anti-aircraft guns, 700 search lights, and 100 barrage balloons. This exercise showed a great improvement in the operation of the air defence warning and control system. The contribution made by the OR teams was so apparent that the Air Officer Commander-in-Chief RAF Fighter Command (Air Chief Marshal Sir Hugh Dowding) requested that, on the outbreak of war, they should be attached to his headquarters at Stanmore.

Initially, they were designated the "Stanmore Research Section". In 1941 they were redesignated the "Operational Research Section" when the term was formalised and officially accepted, and similar sections set up at other RAF commands.

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1941 Onward

In 1941, an Operational System Section (ORS) was established in Coastal Command which was to carry out some of the most well-known OR work in World War II.

The responsibility of Coastal Command was, to a large extent, the flying of long-range sorties by single aircraft with the objective of sighting and attacking surfaced U-boats. Amongst the problems that ORS considered were:

1. ***Organization of flying maintenance and inspection:*** Here the problem was that in a squadron each aircraft, in a cycle of approximately 350 flying hours, required in terms of routine maintenance 7 minor inspections lasting 2 to 5 days each and a major inspection lasting 14 days. The question, which needs to be answered, was how can flying and maintenance be organized to make best use of squadron resources?
2. ***Comparison of aircraft type:*** The problem was one of deciding, for a particular type of operation, the relative merits of different aircraft in terms of factors such as: miles flown per maintenance man per month; lethality of load; length of sortie; chance of U-boat sighting; etc.
3. ***Improvement in the probability of attacking and killing a U-boat:*** In early 1941 the attack kill probability was 2% to 3%. It is in this area that the greatest contribution was made by OR in Coastal Command.

Although scientists had been involved in the hardware side of warfare i.e. designing better planes, bombs, tanks, etc. scientific analysis of the operational use of military resources had never taken place in a systematic fashion before the World War II. Military personnel were simply not trained to undertake such analysis.

By the end of the war OR was well established in the armed services both in the UK and in the USA.

Following the end of the war OR took a different course in the UK as opposed to in the USA. In the UK many of the distinguished OR workers returned to their original peacetime disciplines. As such OR did not spread particularly well, except for a few isolated industries like iron, steel and coal. Whereas, in the USA, OR spread to the universities to begin systematic training in OR.

Did u know? The importance of Operations Research in India was first felt by Prof. Mahalanobis when he used OR techniques in the formulation of the second five year plan for forecasting the trend in demand and availability of resources. Planning commission made use of OR techniques for planning the optimum size of the Caravelle fleet of Indian Airlines

1.5 Applications of Operations Research

To give an idea of the application areas in which OR can be applied, given below are abstracts from papers on OR projects that have been reported in the literatures. All these projects are drawn from journals published in recent past.

Note that, at this stage of knowledge of OR, you will probably not understand every aspect of these abstracts but you should have a better understanding of them by the end of the course.

- ***Managing consumer credit delinquency in the US economy: a multi-billion dollar management science application:*** GE Capital provides credit card services for a consumer credit business exceeding $12 billion in total outstanding dollars. Its objective is to optimally manage delinquency by improving the allocation of limited collection resources to

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maximize net collections over multiple billing periods. GE developed a probabilistic account flow model and statistically designed programmes to provide accurate data on collection resource performance. A linear programming formulation produces optimal resource allocations that have been implemented across the business.

- ***Control of the water distribution system under Irrigation scheme in Malaysia:*** A linear programming optimizations model was developed and adapted for daily operating decisions that would provide for a proper control of the water distribution system in real-time for a Irrigation scheme in Malaysia. Water orders are used as input in the optimizations model and the water ordering programme employs a simplified routing method to obtain a solution to the unsteady state condition. The routed flow is then optimized by minimizing the deviation between gate releases and water order demands of the users using the linear programming technique. Physical limitations of the system and water volume balance are used as constraints.

  The results showed that the model is capable of providing the desired operating plan with a more stable flows and manageable releases. By following the models predictions, all the gate demands are met while stable flows in canal reaches are maintained and canal water depths are kept within the desired target levels.

- ***Formulating insurance polices by Life Insurance Company in India:*** LIC uses OR to decide on the premium rate for it's various policies and also how best the profits could be distributed in the case of profit policies.

- ***Application of OR for optimum utilization of urban facilities:*** Increasingly, citizens are demanding more urban services, by type, quantity, and quality. The resulting pressure, between the demands for more and better services, on the one hand, and decreased revenue, on the other, has created a strong need for improved management decision making in urban services.

  Thus, OR is widely used for effective and efficient allocation or deployment of the resources of urban service systems, including personnel, equipment, and various service-improving technologies.

An important consequence of the application of OR to a wide variety of problems is that a small set of problem types has been identified which accounts for most problems. Because of the frequent recurrence of these problems, prototype techniques have been developed for modeling them and for deriving solutions from these models. Prototype applications include:

Forecasting: Using time series analysis to answer typical questions such as, how big will demand for products be? What are the sales patterns? How will this affect profits?

Finance & Investment: How much capital do we need? Where can we get this? How much will it cost?

Manpower Planning & Assignment: How many employees do we need? What skills should they have? How long will they stay with us?

Sequencing & Scheduling: What job is most important? In what order should we do jobs?

Location, Allocation, Distribution & Transportation: Where is the best location for operation? How big should facilities be? What resources are needed? Are there shortages? How can we set priorities?

Reliability & Replacement Policy: How well is equipment working? How reliable is it? When should we replace it?

Inventory Control and Stock out: How much stock should we hold? When do we order more? How much should we order?

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Project Planning and Control: How long will a project take? What activities are most important? How should resources be used?

Queuing and Congestion: How long are queues? How many servers should we use? What service level are we giving?

This broad range of potential applications and a wide variety of OR techniques, which can be selected and combined for a multi-disciplinary approach, work together to make this course a dynamic and exciting one.

Some of the application areas of management where OR is applied is presented below:

- ***Finance, Budgeting and investments***
  - Cash flow analysis,
  - Capital requirement,
  - Investment analysis,
  - Dividend policies,
  - Credit policies,
  - Portfolio analysis, etc.
- ***Purchasing & procurement***
  - Quantity and timing of purchase,
  - Bidding policies,
  - Replacement policies, etc.
- ***Production Management***
  - Production scheduling,
  - Physical distribution,
  - Inventory control,
  - Manufacturing and facilities planning,
  - Maintenance policies,
  - Product-mix planning, etc.
- ***Marketing Management***
  - Product selection and competitive actions,
  - Advertising strategy,
  - Market research, etc.
- ***Personnel Management***
  - Recruitment policies,
  - Selection procedure,
  - Salary structure,
  - Bonus schemes,
  - Scheduling of training programmes, etc.

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- *Research and Development*
  - Control of R&D projects,
  - Reliability and evaluation of alternative designs,
  - Determination of time and cost, etc.

Self Assessment

State true or false:

7. There is no unique set of problems which can be solved using OR models or techniques.
8. OR methods and techniques are only applicable to big enterprises.
9. OR can be applied in almost all disciplines of management such as marketing, finance, HR, research and development.

Task Enlist the OR concepts that you applied while participating in managerial activities in the organizations that you have been a part of in the past.

1.6 Models of Operations Research

When a problem or process under investigation is simplified and represented with its typical features or characteristics, it is called as a model. The word 'model' has several meanings. All of which are relevant to Operations Research. For example, a model can act as a substitute for representing reality, such as small scale model locomotive or may act as some sort of idealization, like a model plan for recruitment procedure, etc. Constructing a model helps in bringing the complexities and possible uncertainties into a logical framework required for comprehensive analysis. In fact the model acts as a vehicle in arriving at a well-structured view of reality. An array of models can be seen in various business areas or industrial activities. The most relevant for our study purpose are portrayed below:

1. ***Physical Models:***
   (a) Iconic Models
   (b) Analogue Models
2. ***Symbolic Models:***
   (a) Verbal Models
   (b) Mathematical Models:
      (i) Deterministic Models
      (ii) Probabilistic Models
3. Combined Analogue and Mathematical Models
4. Function Models
5. Heuristic Models

1. ***Physical Models:*** To deal with specific types of problems, models like diagrams, charts, graphs and drawings are used, which are known as "Physical Models'. The schematic way of representation of significant factors and interrelationship may be in a pictorial form

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help in useful analysis. Moreover, they help in easy observation, description and prediction. However, they over rule any manipulation.

There are two types of Physical Models:

(i) *Iconic Models:* An image or likeness of an object or process is Icon. These models represent the system as it is by scaling it up or down. Even though use of these models in the area of management appears to be narrow, their usefulness is seen in the field of engineering and science. For example,

(a) In the field of R&D, prototype of the product is developed and tested to know the workability of the new product development.

(b) Photographs, portraits, drawings are the good example of iconic types. These models help in testifying the samples thus avoiding fullscale designing and probable loss.

(ii) *Analogue Models:* These models are closely associated with iconic models. However, they are not the replicas of system or process. The analogue, in constructing these models, help in analyzing the issues and forces which are in the system or process. Because, these models use 'one set of properties' which is 'analogous to another set of properties.' For example, kids, toys, rail-road models, etc.

(2) ***Symbolic Models:*** Symbolic Models use letters, numbers, figures to represent decision variables of the system. There are two types of Symbolic Models—Verbal Models and Mathematical Models.

(a) *Verbal Models:* These models describe a situation in written or spoken language. Written sentences, books, etc., are examples of a verbal model.

(b) *Mathematical Models:* Mathematical symbols are used to represent a problem or a system under these types of models. Rules of mathematics enable the builder to make the models more abstract and precise. There are two types of Mathematical Models—Deterministic Models and Probabilistic Models.

(i) **Deterministic Models:** The exact statement of variables and their relationships are made under these models. The coefficients used for the mathematical formulation are known and are constant with certainty. So, to say, with a given set of data the answer will always be the same. For instance, determination of the break-even sales volume (BEP), the volume where the total cost equals the total sales revenue earned pertaining to a product.

The sales revenue line is determined by using an equation.

$$TSR = (SV)(PU)$$

where, TSR = Total Sales revenue

SV = Sales Volume

PU = Price per Unit

The total cost line is represented as:

$$TC = TFC + SV(VC)$$

TC = Total Cost

TFC = Total Fixed Cost

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SV = Sales Volume

VC = Variable Cost.

The break-even point occurs when

TSR = TC or (SV) (PU) = TFC + SV (VC)

Hence, the volume (BEP) is determined by

$$BEP = \frac{FC}{SP - VC}$$ – VC represents the contribution to fixed cost.

(ii) **Probabilistic Models:** The risk involved and the state of uncertainty are covered by these models. The decision variables take the form of a probability distribution and can assume more than single values. In the presence of risk and uncertainty, these models tend to yield different answers every time when attempted to. For example, uncertainty over acquisition of raw material to execute customer orders during a certain period. The purchaser has to consider both the sale and the delivery timing of the orders as they are variables. A probability distribution can be developed for the instant period for both sales delivery timings. However, the optimum selection is adhered to in accordance with the demand of the situation.

(3) ***Combined Analogue and Mathematical Models:*** When analogue models are interpreted with the use of mathematical symbols, they can be termed as physical-symbolic models. For instance, decision-makers tend to use mathematical symbols to represent their sales or profit figures.

(4) ***Function Models:*** Models which are used to represent a group of functions performed, are called function models.

Example: A monthly plan of processes to be carried on, a list of layouts, a calendar of events are the common examples of a functional model.

(5) ***Heuristic Models:*** When intuition guides a problem-solver to find solutions, heuristic models are developed. Even though he would not be able to find an optimum solution to the problem, with his past experience he arrives at the most advantageous solution. However, it depends on how intuitive and creative the decision-maker is.

Caution Models do not, and cannot represent every aspect of reality because of the innumerable and changing characteristics of the real-life problems to be represented. Instead, a model is the simplified representation or abstraction of reality.

Task Give real world examples of following OR models:

1. Analogue models
2. Symbolic models
3. Probabilistic model
4. Deterministic model

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Self Assessment

Fill in the blanks:

10. OR approach is typically based on the use of ……. model.
11. Monthly plan of processes to be carried on is an example of ……..model.
12. There are two types of …….models—Verbal Models and Mathematical Models.

1.7 Summary

Operations Research (OR) is an analytical method of problem-solving and decision-making that is useful in the management of organizations. Thus it is also known as a mathematical or scientific analysis of a process or operation, used in making decisions. In other words, OR represents the study of optimal resource allocation. The aim of OR is to provide rational basis for decision making by seeking to understand and structure complex situations, and to utilize this understanding to predict system behavior and improve system performance.

The tools of OR are not from any one discipline, rather mathematics, statistics, economics, engineering, psychology etc, have contributed to this newer discipline of knowledge. OR is the application of the methods of science to complex problem arising in the direction and management of large system of men, machines, materials and money in industry, business, and government of defense. The mathematical model can be deterministic or probabilistic in nature. In deterministic model the parameters are considered to be defined and the outcome related to a certain course of action is known with certainty. A problem can be modeled in differing ways, and the choice of the appropriate model may be crucial to the success of the OR project.

1.8 Keywords

Analogue Models: Models which look like the real situation but represent and behave like a system under study.

Operations Research: Operations Research is a scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources.

Physical Models: To deal with specific types of problems, models like diagrams, charts, graphs and drawings are used, which are known as "Physical Models".

Symbolic models: Models which use symbols and functions to represent variables and their relationships to describe the properties of the system.

Verbal Models: These models describe a situation in written or spoken language. Written sentences, books, etc., are examples of a verbal model.

1.9 Review Questions

1. What is Operations Research?
2. Give the historical background of Operations Research.
3. Give any three definitions of Operations Research and explain.
4. Discuss the significance and scope of Operations Research in Modern Management.
5. Define Operations Research and discuss its importance in decision-making.

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6. What is Operations Research? What areas of Operations Research have made a significant impact on decision-making process? Why is it important to keep an open mind in utilizing Operations Research techniques?
7. Comment on the following statements:
   (i) Operations Research is the art of winning war without actually fighting it.
   (ii) Operations Research is the art of finding bad answers where worse answers exist.
8. State the different types of models in Operations Research.
9. "Operations Research replaces management by personality". Discuss.
10. What are the situations where Operations Research techniques will be applicable?
11. 'Quantitative techniques complement the experience and judgement of an executive in decision making. They do not and cannot replace it'. Discuss.
12. 'Operations Research is more than a quantitative analysis of the problem'. Comment.
13. 'Operations Research advocates a system approach and is concerned with optimization. It provides a quantitative analysis for decision-making'. Comment.
14. Give the description of the following basic types of models:
   (i) Iconic
   (ii) Analogue
   (iii) Mathematical

Answers: Self Assessment

1.	(a)	2.	(a)
3.	(c)	4.	Capital Budgeting
5.	Production systems	6.	Operation Researchers
7.	True	8.	False
9.	True	10.	Mathematical
11.	Functional	12.	Symbolic

1.10 Further Readings

Books

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Unit 2: Linear Programming Problems

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2.12 Further Readings

Objectives

After studying this unit, you will be able to:

- Understand what is linear programming
- Locate areas of application with its scope
- Know how to formulate LP models and use graphical procedure to solve them

Introduction

Each and every organization aspires for optimal utilization of its limited scarce resources like men, money, materials, machines, methods and time to reach the targets. The results are generally measured in terms of profits, losses, return on money invested, etc. To achieve these results, the decision-maker has to have thorough knowledge about the tasks or jobs and the relationships among them. Among the popular techniques of Operations Research, Linear Programming deserves mention because it is one of the widely used techniques. And it is a deterministic model. In other words, Linear Programming is one of the important Operations Research tools used to allocate scarce resources in an optimal way so that the allocator can optimize the results either by maximizing the profits or minimizing the costs. The credit of innovating this technique goes to George B. Dantzig. He innovated this technique while he was working for U.S. Air force during World War II, 1947. Initially, this technique was used to solve tough logistic problems like assignment and transportation but instantly the application of this technique has spread to almost every functional area of management, production planning and control, personnel management, advertising and promotion.

Notes

Linear Programming decisions are made obviously under certainty conditions i.e., when the existing situation and the variables are known. The results obtained will be either optimal or nearly optimal. It even helps in cross verification of the results obtained through the process of mere intuition and the one arrived at with the use of Linear Programming technique while an optimum solution is being anticipated.

The general Linear Programming Problem calls for optimizing (maximizing/minimizing) a linear function for variables called the 'objective function' subject to a set of linear equations and/or inequalities called the 'constraints or restrictions.'

2.1 Basic Terminology

The word 'linear' is used to describe the relationship among two or more variables which are directly or precisely proportional.

Programming' means the decisions which are taken systematically by adopting alternative courses of action.

Basic Requirements and their Relationships

1. ***Decision Variables and their Relationships:*** The decision variable refers to any candidate (person, service, projects, jobs, tasks) competing with other decision variables for limited resources. These variables are usually interrelated in terms of utilization of resources and need simultaneous solutions, i.e., the relationship among these variables should be linear.
2. ***Objective Function:*** The Linear Programming Problem must have a well defined objective function to optimize the results. For instance, minimization of cost or maximization of profits. It should be expressed as linear function of decision variables ($Z = X_1 + X_2$, where Z represents the objective, i.e., minimization/maximization, X_1 and X_2 are the decision variables directly affecting the Z value).
3. ***Constraints:*** There would be limitations on resources which are to be allocated among various competing activities. These must be capable of being expressed as linear equalities or inequalities in terms of decision variables.
4. ***Alternative Courses of Action:*** There must be presence of alternative solutions for the purpose of choosing the best or optimum one.
5. ***Non-Negativity Restrictions:*** All variables must assume non-negative values. If any of the variable is unrestricted in sign, a tool can be employed which will enforce the negativity without changing the original information of a problem.
6. ***Linearity and Divisibility:*** All relationships (objective function and constraints) must exhibit linearity i.e., relationship among decision variables must be directly proportional. It is assumed that decision variables are continuous, i.e., fractional values of variables must be permissible in obtaining the optimum solution.
7. ***Deterministic:*** In Linear Programming it is assumed that all model coefficients are completely known. For example: profit per unit.

2.2 Application of Linear Programming

LP is a widely used technique of OR in almost every decision of a business and management. However, Linear Programming is exclusively used in the following areas:

1. Production Management

Notes

2. Personnel Management
3. Financial Management
4. Marketing Management

Apart from these areas it is used in agricultural operations, farm management and military problems.

1. ***Production Management:*** In the area of production management, Linear Programming is used in the field of:
   - Product planning
   - Research and development
   - Product portfolio management
   - Line expansion and contraction decision
   - Longevity of product life cycle.
2. ***Personnel Management:*** In this area, LP is used in the field of:
   - Recruitment and staffing decisions
   - Wage or salary management
   - Job evaluation and allocation
   - Employee benefits and welfare
   - Overtime and related decisions.
3. ***Financial Management:*** In this area, LP is used in the field of:
   - Portfolio decision
   - Profit planning
   - Alternative capital investment decisions
   - Investment on inventories
   - Allocation of funds to developmental activities.
4. ***Marketing Management:*** In the area of marketing management, it is used in the field of:
   - Media planning and selection
   - Travelling salesman problem
   - Product development
   - Ad and Pro budget
   - Marketing mix decisions.

2.3 Advantages and Limitations of Linear Programming

The LP technique has several advantages and at the same time, it is not free from unmixed blessings. They have been put into T-type classification as under:

Notes

Table 2.1: T-type Classification of Advantages and Limitations

Advantages	Limitations
1. It helps in proper and optimum utilization of the scarce resources	1. The treatment of variables having non-linear relationships is the greatest limitation of this LP
2. It helps in improving the quality of the decisions.	2. It can come out with non-integer solutions too, which would be many a times meaningless.
3. With the use of this technique, the decision-maker becomes more objective and less subjective.	3. It rules out effect of time and uncertainty conditions.
4. It even helps in considering other constraints operating outside the problem.	4. Generally, the objective set will be single and on the contrary, in the real life, there might be several objectives.
5. Many a times it hints the manager about the hurdles faced during the production activities.	5. Large-scale problems tend to be unaccomodative to solve under LP

2.4 Formulation of LP Models

Linear Programming Family

The family of LP consists of:

1. Formulation of Linear Programming Problems. (LPP)
2. LP – Graphical Solutions
3. LP – Simplex Solutions
4. LP – Assignment Problems
5. LP – Transportation Problems.

Steps for Formulating LPP

1. Identify the nature of the problem (maximization/minimization problem).
2. Identify the number of variables to establish the objective function.
3. Formulate the constraints.
4. Develop non-negativity constraints.

Example: A firm manufactures 2 types of products A & B and sells them at a profit or ₹ 2 on type A & ₹ 3 on type B. Each product is processed on 2 machines G & H. Type a requires 1 minute of processing time on G and 2 minutes on H. Type B requires one minute on G & 1 minute on H. The machine G is available for not more than 6 hrs. 40 mins., while machine H is available for 10 hrs. during any working day. Formulate the problem as LPP.

Solution:

Let x_1 be the no. of products of type A

x_2 be the no. of products of type B

Notes

Table 2.2: Showing the Time (minutes) Available and the Profit

Machines	Time on Products (mins.)		Total time available (in minutes)
	Type A	Type B	
G	1	1	400
H	2	1	600
Profit Per Unit	₹ 2	₹ 3	

Since the profit on type A is ₹ 2 per product, $2x_1$ will be the profit on selling x_1 units of type A.

Similarly $3x_2$ will be the profit on selling x_2 units of type B.

Hence the objective function will be,

Maximize 'Z' = $2x_1 + 3x_2$ is subject to constraints.

Since machine 'G' takes one minute on 'A' and one minute on 'B', the total number of minutes required is given by $x_1 + x_2$. Similarly, on machine 'H' $2x_1 + x_2$. But 'G' is not available for more than 400 minutes. Therefore, $x_1 + x_2 \leq 400$ and H is not available for more than 600 minutes, therefore, $2x_1 + x_2 \leq 600$ and $x_1, x_2 \geq 0$, i.e.,

$$x_1 + x_2 \leq 400 \qquad \text{(Time availability constraints)}$$

$$2x_1 + x_2 \leq 600$$

$$x_1, x_2 \geq 0 \qquad \text{(Non-negativity constraints)}$$

Example: A company produces 2 types of cowboy hats. Each hat of the first type requires twice as much labour time as the second type. The company can produce a total of 500 hats a day. The market limits the daily sales of first and second types to 150 and 250 hats. Assuming that the profits per hat are ₹ 8 per type A and ₹ 5 per type B, formulate the problem as Linear Programming model in order to determine the number of hats to be produced of each type so as to maximize the profit.

Solution:

Let x_1 be the no. of hats of type A.

x_2 be the no. of hats of type B.

$8x_1$ is the total profit for hats of type A.

$5x_2$ is the total profit for hats of type B.

Hence, objective function will be equal to

Maximise $\quad$ 'Z' = $8x_1 + 5x_2$ $\quad$ (Subject to constraints)

$$2x_1 + x_2 \leq 500 \qquad \text{(Labour time for total production)}$$

$$x_1 \leq 150 \qquad \text{(No. of hats of type A to be sold)}$$

$$x_2 \leq 250 \qquad \text{(No. of hats of type B to be sold)}$$

$$x_1, x_2 \geq 0 \qquad \text{(Non-negativity constraints)}$$

Notes

Example: A firm can produce 3 types of cloth say A, B and C. Three kinds of wool are required for it say red, green and blue. One unit length of type A cloth needs 2 metres of red wool and 3 metres of blue wool. One unit length of B type cloth needs 3 metres of red wool, 2 metres of green wool and 2 metres of blue wool; and 1 unit length of type C cloth needs 5 metres of green wool and 4 metres of blue wool. The firm has a stock of 8 metres of red wool, 10 metres of green wool and 15 metres of blue wool, it is assumed that the income obtained from one unit length of type A cloth is ₹ 3, of B ₹ 5 and of C ₹ 4.

Determine how the firm should use the available material so as to maximize the income from the finished cloth. Formulate the above problem as LPP.

Solution:

Let x_1 be the type of cloth A

x_2 be the type of cloth B

x_3 be the type of cloth C

Therefore $3x_1$ is the profit for type A cloth

$5x_2$ is the profit for type B cloth

$4x_2$ is the profit for type C cloth.

Materials	Cloth			Max. Material Available
	A	B	C	
Red	2	3	--	8
Blue	3	2	4	15
Green	--	2	5	10
Profit per unit (₹)	3	5	4	

The Objective function is given by

Maximize $'Z' = 3x_1 + 5x_2 + 4x_3$ (Subject to constraints)

$2x_1 + 3x_2 \leq 8$ (Material Constraint)

$3x_1 + 2x_2 + 4x_3 \leq 15$ (Material Constraint)

$2x_2 + 5x_3 \leq 10$ (Material Constraint)

$x_1, x_2, x_3 \leq 0$ (Non-negativity constraints)

Example: A firm manufactures 3 types of products A, B and C. The profits are ₹ 3, ₹ 2 and ₹ 4 respectively. The firm has 2 machines and below is the required processing time in minutes for each machine from product.

Machine	Product		
	A	B	C
C	4	3	5
D	3	2	4

Machine C and D have 2,000 and 2,500 machine minutes respectively. The firm must manufacture 100 A's, 200 B's and 50 C's but not more than 150 A's.

Formulate an LPP to maximize the profit.

Notes

Solution:

Let x_1 be the type of product A

x_2 be the type of product B

x_3 be the type of product C

Therefore the objective function will be,

Maximize $'Z' = 3x_1 + 2x_2 + 4x_3$ (Subject to constraints)

$$4x_1 + 3x_2 + 5x_3 \leq 2{,}000$$

$$3x_1 + 2x_2 + 4x_3 \leq 2{,}500 \quad \text{(Machine hour constraints)}$$

$$x_1 \leq 150 \text{ or } 0 \leq x_1 \leq 150$$

$$0 \leq x_2 \leq 200 \quad \text{(Production constraint)}$$

$$0 \leq x_3 \leq 50$$

$$x_1, x_2, x_3 \leq 0 \quad \text{(Non-negativity constraints)}$$

Self Assessment

Multiple Choice Questions:

1. Linear Programming is a
   (a) constrained optimization technique
   (b) Technique for economic allocation of limited resource
   (c) Mathematical technique
   (d) All of the above
2. A constraint in a LP model restricts
   (a) Value of an objective function (b) Value of a decision variable
   (c) Use of the available resources (d) All of the above
3. Constraints of an LP model represents
   (a) Limitations
   (b) Requirements
   (c) Balancing limitations and requirements
   (d) All of the above

2.5 Maximization Cases with Mixed Constraints

Example: The manager of an oil-refinery must decide on the optimal mix of 2 possible blending processes, of which the input and output per production run are given as follows:

Process	Input (units)		Output (units)	
	Crude 'A'	Crude 'B'	Gasoline X	Gasoline Y
I	5	3	5	8
II	4	5	4	4

Notes

The maximum amount available of crude A and B are 200 units and 150 units respectively. Market requirements show that at least 100 units of gasoline X and 80 units of gasoline Y must be produced. The profit per production run from process I and process II are ₹ 300 and ₹ 400 respectively. Formulate the above problem as LPP.

Solution:

Let x_1 represent process I

x_2 represent process II

Therefore, 300 x1 represent profit on process I

$400x_2$ represent profit on process II

Hence, the objective function is given by,

Maximize $'Z' = 300x_1 + 400x_2$ (Subject to constraints

$5x_1 + 4x_2 \leq 200$

$3x_1 + 5x_2 \leq 150$ (Crude Oil constraints)

$5x_1 + 4x_2 \geq 100$

$8x_1 + 4x_1 \geq 80$ (Gasoline constraints)

$x_1, x_2 \leq 0$ (Non-negativity constraints)

Example: The management of xyz corporation is currently faced with the problem of determining its product mix for the coming period. Since, the corporation is one of the few suppliers of transformers for laser cover units, only liberal sales ceilings are anticipated. The corporation should not plan on selling transformers more than 200 units of A type, 100 units of B type and 180 units of C type. Contracts call for production of at least 20 units of A type and 70 units of C type. Within these bounds, management is free to establish units production schedules. These are subject to the capacity of the plant to produce without overtime. The production times prevail.

Product	Production Hours per unit				Unit Profit (₹)
	Dept. I	Dept. II	Dept. III	Dept. IV	
A	0.10	0.06	0.18	0.13	10
B	0.12	0.05	---	0.10	12
C	0.15	0.09	0.007	0.08	15
Available Hours	36	30	37	38	

Formulate this as an LPP so as to maximize the total profit.

Solution:

Let x_1 be the units of A type.

x_2 be the units of B type.

x_3 be the units of C type.

Therefore $10x_1$ be the profit of A type.

$12x_2$ be the profit of B type.

$15x_3$ be the profit of C type.

Notes

Hence, the objective function is given by,

Maximize $'Z' = 10x_1 + 12x_2 + 15x_3$ (Subject to constraints)

$0.10x_1 + 0.12x_2 + 0.15x_3 \leq 36$

$0.12x_1 + 0.05x_2 + 0.09x_3 \leq 30$ (Production hours constraints)

$0.18x_1 + 0 + 0.07x_3 \leq 37$

$0.13x_1 + 0.10x_2 + 0.08x_3 \leq 38$

$x_1 \leq 200$

$x_2 \leq 100$ (Sales constraints)

$x_3 \leq 180$

$x_1 \geq 20$ (Production constraints)

$x_2 \geq 70$

OR

$20 \leq x_1 \leq 200$

$70 \leq x_3 \leq 180$ (Sales and Production constraints)

$x_2 \leq 100$

$x_1, x_2, x_3 \leq 0$ (Non-negativity constraints)

Example: An advertising company wishes to plan advertising campaign in 3 different media, television, radio and magazine. The purpose of advertising is to reach as many potential consumers as possible. Results of a marketing study are given below:

Particulars	Television		Radio	Magazines
	Prime Day	Prime Time		
Cost of an Advertising Unit (₹)	40,000	75,000	30,000	15,000
No. of Potential Customers Reached per Unit	4,00,000	9,00,000	5,00,000	2,00,000
No. of Women Customers Reached per Unit	3,00,000	4,00,000	2,00,000	1,00,000

The company doesn't want to spend more than ₹ 8,00,000 on advertising. It further requires that:

a. At least 2 million exposures take place among women.

b. Advertising on television be limited to ₹ 5,00,000.

c. At least 3 advertising units can be bought on prime day and 2 units prime time.

d. The number of advertising units on radio and magazine should each be between 5 and 10.

Formulate the Linear Programming model in order to maximize the total number of potential customers reached.

Notes

Solution:

Let x_1 represent the advertising on prime day on television

x_2 represent the advertising on prime time on television

x_3 represent the campaign on radio

x_4 represent the campaign on magazine.

Therefore $4,00,000x_1$ represent the potential customers on advertising on prime day on television.

$9,00,000x_2$ represent the potential customers on prime time on television.

$5,00,000x_3$ be the potential customers on advertising on Radio.

$2,00,000x_4$ be the potential customers on advertising in Magazines.

Hence the objective function is given by

Maximum $'Z' = 4,00,000x_1 + 9,00,000x_2 + 5,00,000x_3 + 2,00,000x_4$ (Subject to constraints)

$40,000x_1 + 75,000x_2 + 30,000x_3 + 15,000x_4 \leq 8,00,000$ (Advertising constraint)

$40,000x_1 + 75,000x_2 \leq 5,00,000$ (Advertising on television constraint)

$3,00,000x_1 + 4,00,000x_2 + 2,00,000x_3 + 1,00,000x_4 \leq 2$ Million (No. of women customers constraint)

$x_1 \geq 3$

$x_2 \geq 2$ No. of unit constraints)

$5 \leq x3 \leq 10$

$5 \leq x_4 \leq 10$ (Minimum no. of advertisements allowed constraints)

Therefore $x_1, x_2, x_3, x_4 \leq 0$ (Non-negativity constraints)

Example: A city hospital has the following daily requirements of nurses at the minimal level:

Period	Clock Time (24 hours a day)	Minimal no. of nurses required
1	6 a.m. – 10 a.m.	2
2	10 a.m. – 2 p.m.	7
3	2 p.m. – 6 p.m.	15
4	6 p.m. – 10 p.m.	8
5	10 p.m. – 2 a.m.	20
6	2 a.m. – 6 a.m.	6

Nurses report to the hospital at the beginning of each period and work for 8 consecutive hours. The wants to determine minimal number of nurses to be employed, so that there will be sufficient number of nurses available for each period.

Formulate this as LP model by setting up appropriate constraints and objective function.

Notes

Solution:

Let x_1 be the no. of nurses working during period 1

x_2 be the no. of nurses working during period 2, and

x_3, x_4, x_5 and x_6 be the no. of nurses working during period 3,4,5, and 6 respectively.

Hence, the objective function is given by,

Minimise $\quad$ 'C' = $x_1 + x_2 + x_3 + x_4 + x_5 + x_6$ $\quad$ (Subject to constraints)

$$x_1 + x_2 \geq 2$$
$$x_2 + x_3 \geq 7$$
$$x_3 + x_4 \geq 15$$
$$x_4 + x_5 \geq 8$$
$$x_5 + x_6 \geq 20$$
$$x_6 + x_1 \geq 6$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \quad \text{(Non-negativity constraints)}$$

Notes Steps of linear programming model formulation are summarized as follows:

Step 1: Identify the decision variables

Step 2: Identify the problem data

Step 3: Formulate the constraints

Step 4: Formulate the Objective Function

Task **Formulate the following as LPP**

One of the interesting problems in Linear Programming is that of balanced diet. Dieticians tell us that a balanced diet must contain certain quantities of nutrients such as proteins, minerals, vitamins, etc. Suppose that you are asked to find out the food that should be recommended from a large number of alternative sources of these nutrients, so, that the total cost of food satisfying minimum requirements of balanced diet is the lowest. The medical experts and the dieticians tell us that it is necessary for an adult to consume at least 75 gms. of proteins, 85 gms. of fats and 300 gms. of carbohydrates daily. The following table gives the different items (which are readily available in the market); Item analysis and their respective costs.

Formulate this problem as LPP

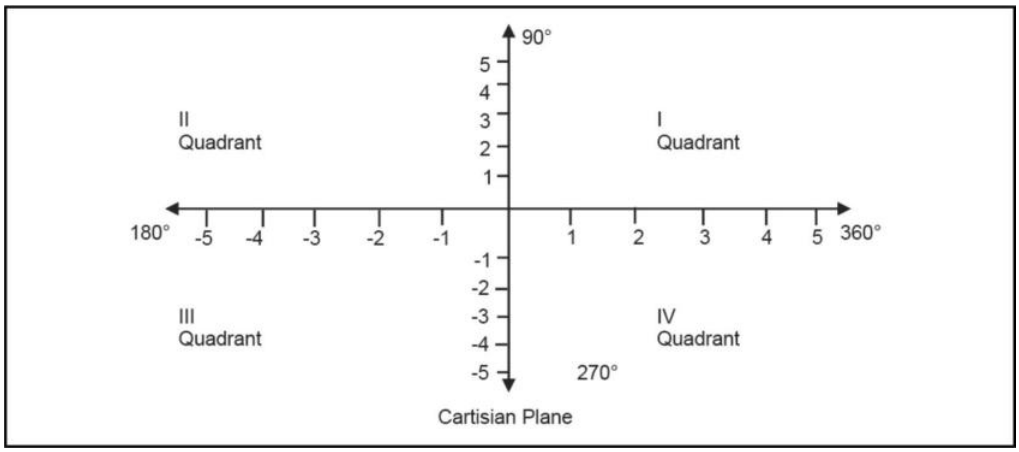
Food	Food volume (per 100 gms.)			Cost/Kg. (₹)
	Proteins	Fats	Carbohydrates	
1	8	1.5	35	1
2	18	15	—	3
3	16	4	7	4
4	4	20	2.5	2
5	5	8	40	1.5
6	2.5	—	25	3
Minimum daily requirements	75	85	300	

Notes

2.6 Graphical Solutions under Linear Programming

Linear programming problems with two variables can be represented and solved graphically with ease. Though in real-life, the two variable problems are practiced very little, the interpretation of this method will help to understand the simplex method.

Following is the portrayal of Cartesian plane.



Steps

1. Consider each inequality constraint as an equation.
2. Plot each equation on the graph as each will geometrically represent a straight line.
3. Plot the feasible region, every point on the line will satisfy the equation on the line.
4. If the inequality constraint corresponding to that line is less than or equal to, then the region below the line lying in the 1st quadrant (as shown in above graph) is shaded (due to non-negativity of variables); for the inequality constraint with greater than or equal to sign, the region above the line in the 1st quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. Hence, it is called feasible region.
5. Identify the co-ordinates of the corner points.
6. Find the 'Z' value by substituting the co-ordinates of corner points to the objective functions.

Example:

Maximize $'Z' = 3x_1 + 5x_2$ (Subject to constraints)

$x_1 + 2x_2 \leq 2{,}000$

$x_1 + x_2 \leq 1{,}500$

$x_2 \leq 600$

$x_1, x_2 \leq 0$ (Refer Problem No.4 in the earlier portion of the unit)

Solution:

Step 1: Convert the inequalities into equalities and find the divisibles of the equalities.

Equation	x_1	x_2
$x_1 + 2x_2 = 2{,}000$	2,000	1,000
$x_1 + x_2 = 1{,}500$	1,500	1,500
$x_2 = 600$	---	600

Notes

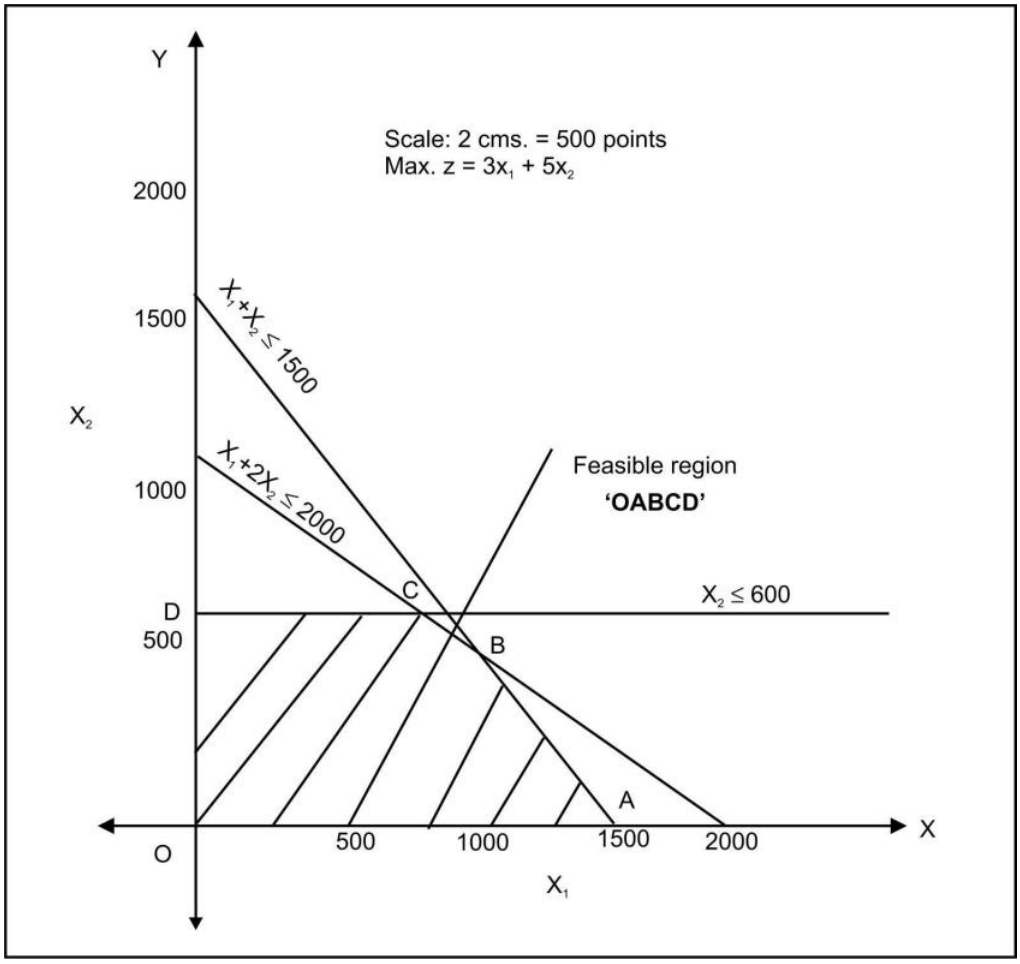
Step 2: Fix up the graphic scale.

Maximum points = 2,000

Minimum points = 600

2 cms = 500 points

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner Points	X_1	x_2
O	0	0
A	1,500	0
B	1,000	500
C	800	600
D	0	600

At 'B': $x_1 + 2x_2 = 2{,}000$ (1)

$x_1 + x_2 = 1{,}500$ (2)

$x_2 = 500$

Notes

Put $x_2 = 500$ in eq. (1),

$x_1 + 2(500) = 2{,}000$

Therefore $x_1 = 2{,}000 - 1{,}000$

Therefore $x_1 = 1{,}000$

At 'C': $x_1 + 2x_2 = 2{,}000$ .....(1)

$x_2 = 600$ .....(2)

Put $x_2 = 600$ in eq. (1),

$x_1 + 2(600) = 2{,}000$

$x_1 = 2{,}000 - 1{,}200$

Therefore $x_1 = 800$

Step 5: Substitute the co-ordinates of corner points into the objective function.

Maximise 'Z' = $3x_1 + 5x_2$

At 'O', Z = 0 + 0 = 0

At 'A', Z = 3 (1,500) + 5 (0) = 4,500

At 'B', Z = 3 (1,000) + 5 (500) = 5,500

At 'C', Z = 3 (800) + 5 (600) = 5,400

At 'C', Z = 3 (0) + 5 (600) = 3,000

Inference

A maximum profit of ₹ 5,500 can be earned by producing 1,000 dolls of basic version and 500 dolls of deluxe version.

Example:

Maximise

'Z' = $2x_1 + 3x_2$ (Subject to constraints)

$x_1 + x_2 \leq 400$

$2x_1 + x_2 \leq 600$

$x_1, x_2 \geq 0$ (Non-negativity constraints)

Solution:

Step 1: Find the divisible points on inequalities

Equation	x_1	x_2
$x_1 + x_2 = 400$	400	400
$2x_1 + x_2 = 600$	300	600

Notes

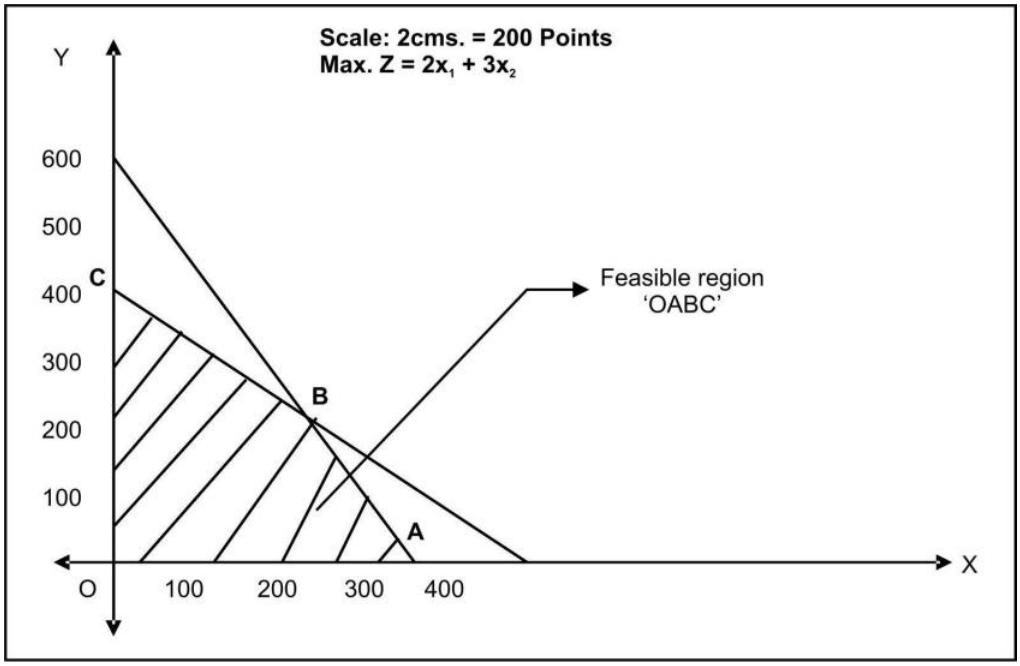
Step 2: Fix up the graphic scale

Maximum points = 600

Minimum points = 300

2 cms. = 200 points

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner Points	X_1	x_2
O	0	0
A	300	0
B	200	200
C	0	400

Step 5: Subscribe the co-ordinates of the corner points into objective function

Maximise $'Z' = 2x_1 + 3x_2$

At 'O', Z = 2(0) + 3(0) = 0

At 'A', Z = 2(300) + 3(0) = 600

At 'B', Z = 2(200) + 2(200)= 1,000

At 'C', Z = 2(0) + 3(400) = 1,200

Inference

A maximum profit of ₹ 1,200 can be earned by producing 400 units of only type B and none of type A.

Notes

Example:

Maximise $'Z' = 8x_1 + 5x_2$ [Subject to constraints]

$x_1 \leq 150$

$x_2 \leq 250$

$2x_1 + x_2 \leq 500$

$x_1, x_2 \geq 0$

Solution:

Step 1: Convert the inequalities into equalities and find the divisibles.

Equation	x_1	X_2
$2x_1 + x_2 = 500$	250	500
$x_1 = 150$	150	-----
$x_2 = 250$	-----	250

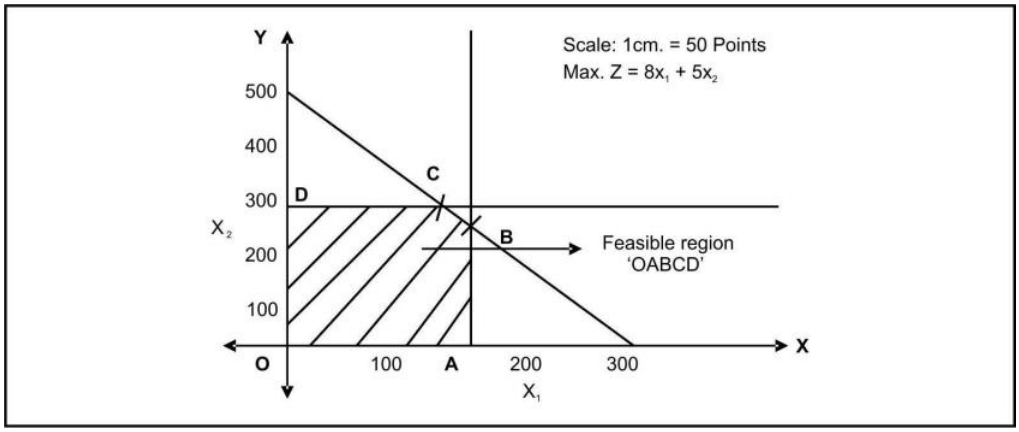
Step 2: Fix up the graphic scale

Maximum points = 500

Minimum points = 150

1 cm. = 50 points

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points.

Corner Points	x_1	X_2
O	0	0
A	150	0
B	150	200
C	125	150
D	0	250

At 'B': $2x_1 + x_2 = 500$ ….(1)

$x_1 = 150$ ….(2)

Notes

Put $x_1 = 150$ in eq. (1)

$2(150) + x_2 = 500$

$x_2 = 500 - 300$

$x_2 = 200$

$2x_1 + x_2 = 2500$ ….(1)

$x_2 = 2500$ ….(2)

Put $x_2 = 2500$ in eq. (1),

$2x_1 + 250 = 500$

$2x_1 = 250$

$x_1 = 125$

Step 5: Substitute the co-ordinates of corner points into the objective function.

Maximise $'Z' = 8x_1 + 5x_2$

At 'O', $Z = 8(0) + 5(0) = 0$

At 'A', $Z = 8(150) + 5(0) = 1,200$

At 'B', $Z = 8(150) + 5(200) = 2,200$

At 'C', $Z = 8(0) + 5(250) = 2,250$

At, 'D' $Z = 8(0) + 5(250) = 1,250$

Inference

Hence, to get a maximum profit of ₹ 2,250, the company has to manufacture 125 units of type 1 cowboy hats and 250 units of type 2 hats.

Example:

Maximise $'Z' = 8,000x_1 + 7,000x_2$ (Subject to constraints)

$3x_1 + x_2 \leq 66$

$x_1 \leq 20$

$x_2 \leq 40$

$x_1 + x_2 \leq 45$

$x_1, x_2 \geq 0$ (Non-negativity constraints)

Solution:

Step 1: Convert the inequalities into equalities and find the divisible.

Equation	x_1	x_2
$3x_1 + x_2 = 66$	22	66
$x_1 = 20$	20	0
$x_2 = 40$	0	40
$x_1 + x_2 = 45$	45	45

Notes

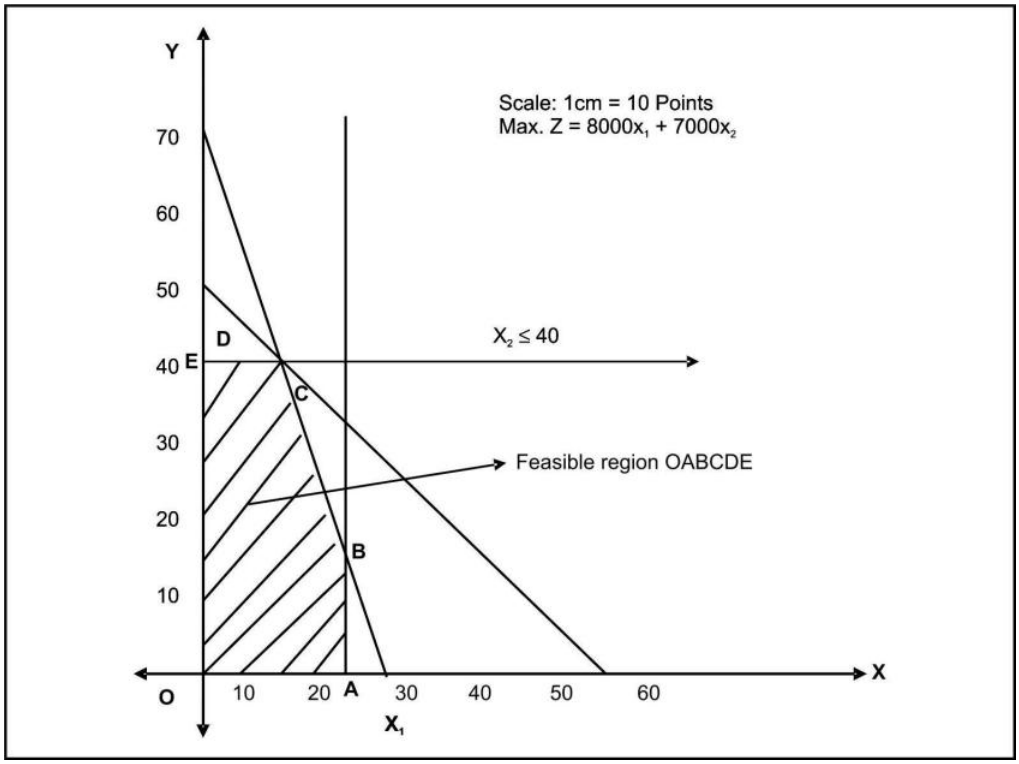
Step 2: Fix up the graphic scale

Maximum points = 66

Minimum points = 20

1 cm. = 10 points

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner points	x1	x2
O	0	0
A	20	0
B	20	6
C	10.5	34.5
D	5	40
E	0	40

At 'C' = $3x_1 + x_2 = 66$ …..(1)

(-) $x_1 + x_2 = 45$ …..(2)

$2x_1 = 21$

Therefore $x_1 = 10.5$

Substituting x_1 10.5 in eq. (2)

Therefore $x_1 + x_2 = 45$

Notes

Therefore $10.5 + x_2 = 45$

$x_2 = 34.5$

At 'B': $x_1 = 20$,

Therefore $3x_1 + x_2 = 66$

Therefore $3(20) + x_2 = 66$

Therefore $x_2 = 66 - 60$

Therefore $x_2 = 6$

Step 5: Substituting the co-ordinates of corner points into objective function.

Maximise 'Z' = $8000x_1 + 7000x_2$

At 'O', Z = 8000(0) + 7000(0) = 0

At 'A', Z = 8000(20) + 7000(0) = 1,60,000

At 'B', Z = 8000(20) + 7000(6) = 2,02,000

At 'C', Z = 8000(10.5) + 7000(34.5) = 3,25,500

At 'D', Z = 8000(5) + 7000(40) = 3,20,000

At 'E', Z = 8000(0) + 7000(40) = 2,80,000

Inference

To maximize the profit, i.e., at ₹ 3,25,500 the company has to manufacture 10,500 bottles of type A medicine and 34,500 bottles of type B medicine.

Task Give the graphical solution for the following LPP

Maximize 'Z' = $0.50x_2 - 0.10x_1$, (Subject to constraints)

$2x_1 + 5x_2 \leq 80$

$x_1 + x_2 \leq 20$

$x_1, x_2 \geq 0$

Caution It is very much essential to locate the solution point of the LPP with respect to the objective function type (max or min). If the given problem is maximization, zma*x* then locate the solution point at the far most point of the feasible zone from the origin and if minimization, Zmin then locate the solution at the shortest point of the solution zone from the origin.

Notes

Self Assessment

Fill in the blanks:

4. Graphical method can be used only if there are ……..decision variables.
5. While solving a LP graphically the area bounded by the constraints is called …….
6. If the given problem is maximization, Z_{max} then locate the solution point at the ……point of the feasible zone from the origin

2.7 Minimization Cases of LP

Example: A rubber company is engaged in producing 3 different kinds of tyres A, B and C. These three different tyres are produced at the company's 2 different plants with different production capacities. In a normal 8 hrs working day plant 1 produces 50, 100 and 100 tyres of A, B and C respectively. Plant 2 produce 60, 60 and 200 tyres of type A, B and C respectively. The monthly demand for tyre A, B and C is 2,500, 3,000 and 7,000 units respectively. The daily cost of operation of plant 1 and 2 is ₹ 2,500 and ₹ 3,500 respectively. Find the minimum number of days of operation per month at 2 different plants to minimize the total costs while meeting the demand.

Solution:

Let x_1 be the daily cost of operation in plant 1

x_2 be the daily cost of operation in plant 2

Minimize $'Z' = 2{,}500x_1 + 3{,}500x_2$ (Subject to constraints)

$$50x_1 + 60x_2 \geq 2{,}500$$

$$100x_1 + 60x_2 \geq 3{,}000$$

$$100x_1 + 200x_2 \geq 7{,}000 \quad \text{(Demand Constraints)}$$

$$x_1, x_2 \geq 0 \quad \text{(Non-negativity constraints)}$$

Step 1: Find the divisible of the equalities.

Equation	x1	x2
$50x_1 + 60x_2 = 2{,}500$	50	41.67
$100x_1 + 60x_2 = 3{,}000$	30	50
$100x_1 + 200x_2 = 7{,}000$	70	35

Step 2: Fix up the graphic scale

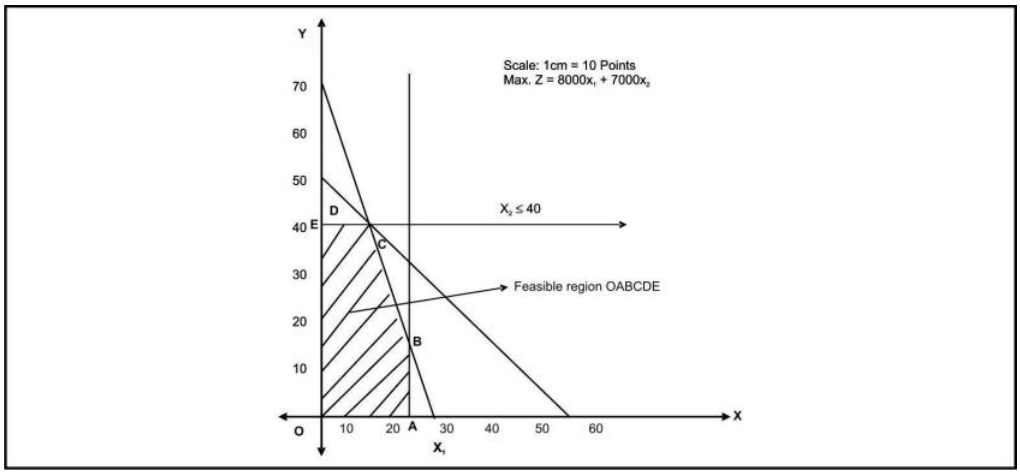
Minimum points = 30

Maximum points = 70

1 cm. = 10 points

Notes

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner Points	x_1	x_2
A	70	0
B	20	25
C	10	33.33
D	0	50

At 'B' $100x_1 + 200x_2 = 7{,}000$ …..(1)

$50x_1 + 60x_2 = 2{,}500$ …..(2)

Divide eq. (1) by 2, we get,

$50x_1 + 100x_2 = 3{,}500$

$50x_2 + 60x_2 = 2{,}500$

$40x_2 = 1{,}000$

Therefore $x_2 = 25$

Put $x_2 = 25$ in eq. (2), $50x_1 + 60(25) = 2{,}500$

$50x_1 + 1{,}500 = 2{,}500$

$50x_1 = 1{,}000$

Therefore $x_1 = 20$

At 'C' = $100x_1 + 60x_2 = 3{,}000$ …..(1)

$50x_1 + 60x_2 = 2{,}500$ …..(2)

$50x_1 = 500$

Therefore $x_1 = 10$

Put $x_1 = 10$ in eq. (2)

$50(10) + 60x_2 = 2{,}500$

$60x_2 = 2{,}000$

Therefore $x_2 = 33.33$

Notes

Step 5: Substitute the co-ordinates of the corner points in the objective.

Minimise 'Z' = $2{,}500x_1 + 3{,}000x_2$

At 'A', Z = 2,500(70) + 3,000(0) = 1,75,000

At 'B', Z = 2,500(20) + 3,000(25) = 1,25,000

At 'C', Z = 2,500(10) + 3,000(33.33) = 1,24,990

At 'D', Z = 2,500(0) + 3,000 (50) = 1,50,000

Inference

Thus, the rubber company can minimize its total cost to ₹ 1,24,990 by producing 10 units of product in plant 1 and 33.33 units in plant 2.

2.8 Cases of Mixed Constraints

Example: A firm that makes products x and y has a total production capacity of 9 tonnes per day, x and y requiring the same production capacity. The firm has a permanent contract to supply at least 2 tonnes of x and 3 tonnes of y per day to another company. Each one of x requires 20 machine hrs. Production time and y requires 50 machine hrs production time. The daily maximum possible number of machine hours available is 360. All the firm's output can be sold, and the profit set is ₹ 80 per tonne of x and ₹ 120 per tonne of y. You are required to determine the production schedule to maximize the firm's profit.

Solution:

Let x_1 be the no. of tonnes of product 'X'

x_2 be the no. tonnes of product 'Y'

Hence, the objective function is given by,

Maximize 'Z' = $80x_1 + 120x_2$ (Subject to constraints)

$20x_1 + 50x_2 \leq 360$ (Machine hour constraint)

$x_1 \geq 2$

$x_2 \geq 3$ (Supply constraints)

$x_1 + x_2 \geq 9$ (Production constraint)

$x_1, x_2 \geq 0$ (Non-negativity constraint)

Step 1: Find the divisibles of the equalities.

Equation	x_1	x_2
$20x_1 + 50x_2 = 360$	18	7.2
$x_1 + x_2 = 9$	9	9
$x_1 = 2$	2	0
$x_2 = 3$	0	3

Step 2: Fix up the graphic scale

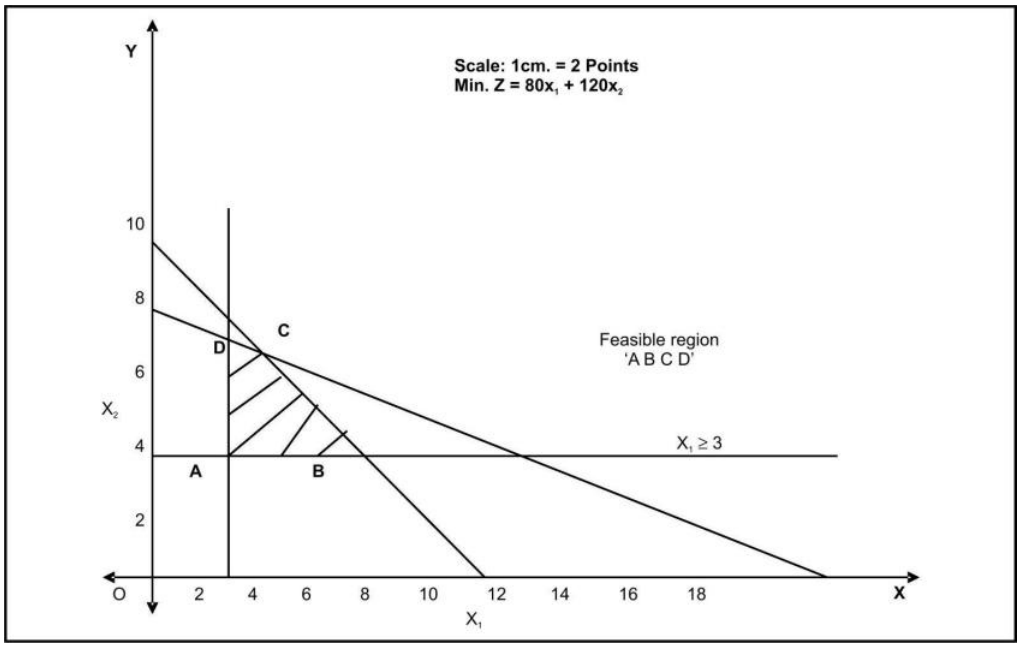
Minimum points = 2

Maximum points = 18

1 cm. = 2 points

Notes

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner Points	x1	x2
A	2	3
B	6	3
C	3	6
D	2	6.4

At 'C': $20x_1 + 50x_2 = 360$ …..(1)

$x_1 + x_2 = 9$ …..(2)

Multiply eq. (2) by 20 and subtract,

$20x_1 + 50x_2 = 360$

$20x_1 + 20x_2 = 180$

$30x_2 = 180$

Therefore $x_2 = 6$

Put $x_2 = 6$ in eq. (2)

$x_1 + 6 = 9$

Therefore $x_1 = 3$

At 'D' $x_1 = 2$ …..(1)

$20x_1 + 50x_2 = 360$ …..(2)

Put $x_1 = 2$ in eq. (2)

$20(2) + 50x_2 = 360$

Notes

$$40 + 50x_2 = 360$$

$$50x_2 = 320$$

Therefore $x_2 = 6.4$

Steps 5: Substitute the values of co-ordinates of the corner points to the objective function.

Maximise $'Z' = 80x_1 + 120x_2$

At 'A', $Z = 80(2) + 120(3) = 520$

At 'B', $Z = 80(6) + 120(3) = 840$

At 'C', $Z = 80(3) + 120(6) = 960$

At 'D' $Z = 80(2) + 120(6.4) = 928$

Inference

The company has to produce 3 tonnes of product x and 6 tonnes of product y in order to maximize the profit.

Example:

Maximise $'Z' = 40x_1 + 60x_2$ (Subject to constraints)

$$2x_1 + x_2 \leq 70$$

$$x_1 + x_2 \geq 40$$

$$x_1 + 3x_2 \leq 90$$

$x_1, x_2 \geq 0$ (Non-negativity constraints)

Solution:

Step 1: Find the divisibles of the equalities.

Equation	x_1	X_2
$2x_1 + x_2 = 70$	35	70
$x_1 + x_2 = 40$	40	40
$x_1 + 3x_2 = 90$	90	30

Step 2: Fix up the graphic scale

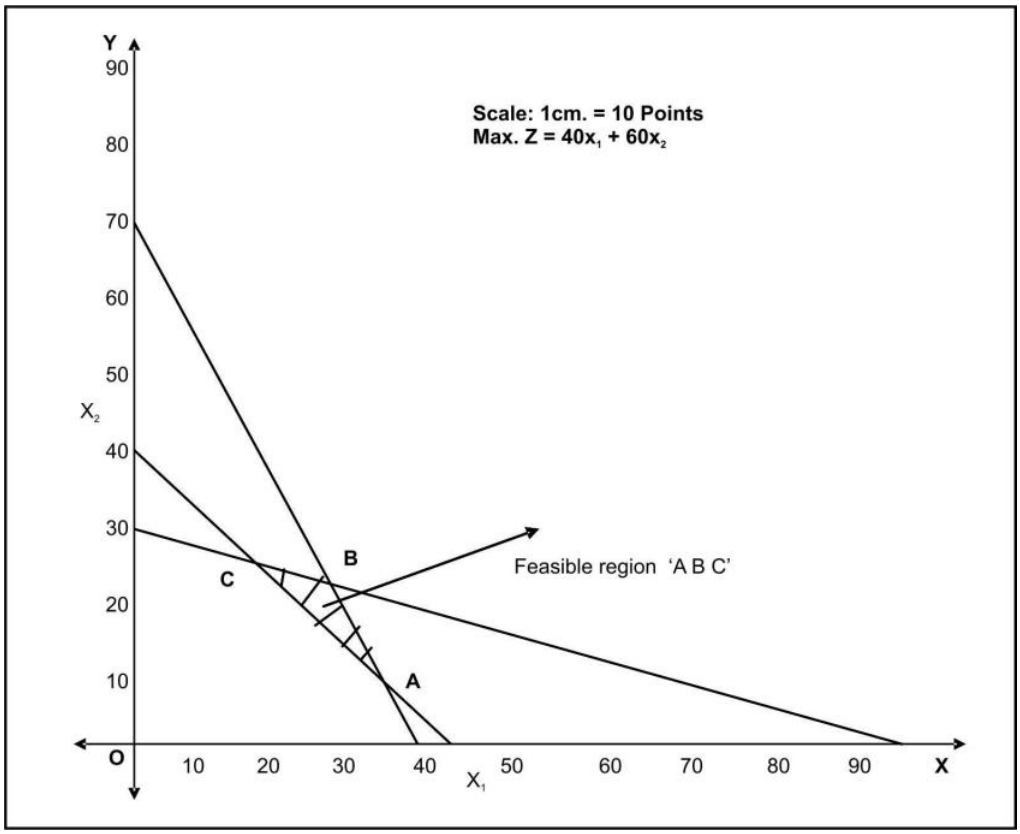
Maximum points = 90

Minimum points = 30

1 cm. = 10 points

Notes

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner Points	x1	x2
A	30	10
B	24	22
C	15	25

At A: $2x_1 + x_2 = 70$ ….(1)

$x_1 + x_2 = 40$ …..(2)

Put $x_1 = 30$ in eq. (2),

$30 + x_2 = 40$

Therefore $x_2 = 10$

At 'B' = $x_1 + 3x_2 = 90$ …..(1)

$2x_1 + x_2 = 70$ …..(2)

Multiply eq. (1) by 2 and subtract,

$2x_1 + 6x_2 = 180$

$2x_1 + x_2 = 70$

$5x_2 = 110$

Notes

Therefore $x_2 = 22$

Put $x_2 = 22$ in eq. (1),

$x_1 + 3(22) = 90$

$x_1 = 90 - 66$

Therefore $x_1 = 24$

At 'C': $x_1 + x_2 = 40$ …..(1)

$x_1 + 3x_2 = 90$ …..(2)

$-2x_2 = -50$

Therefore $x_2 = 25$

Put $x_2 = 25$ in eq. (1),

$x_1 + 25 = 40$

$x_1 = 15$

Step 5: Substitute the co-ordinates of the corner points to the objective function

Maximise 'Z' = $40x_1 + 60x_2$

At 'A', Z = 40(30) + 60(10) = 1,800

At 'B', Z = 40(24) + 60(22) = 2,280

At 'C', Z = 40(15) + 60(25) = 2,100

Inference

Maximum profit can be obtained by producing 24 units of product A and 22 units of product B.

Example:

Maximise 'Z' = $7x_1 + 3x_2$ (Subject to constraints)

$x_1 + 3x_2 \geq 3$

$x_1 + x_2 \leq 4$

$x_1 \leq 5/2$ or 2.5

$x_2 \leq 3/2$ or 1.5

$x_1, x_2 \geq 0$ (Non-negativing constraints)

Solution:

Step 1: Find the divisibles of the equalities

Equation	x_1	x_2
$x_1 + 3x_2 = 3$	3	1
$x_1 + x_2 = 4$	4	4
$x_1 = 2.5$	2.5	0
$x_2 = 1.5$	0	1.5

Notes

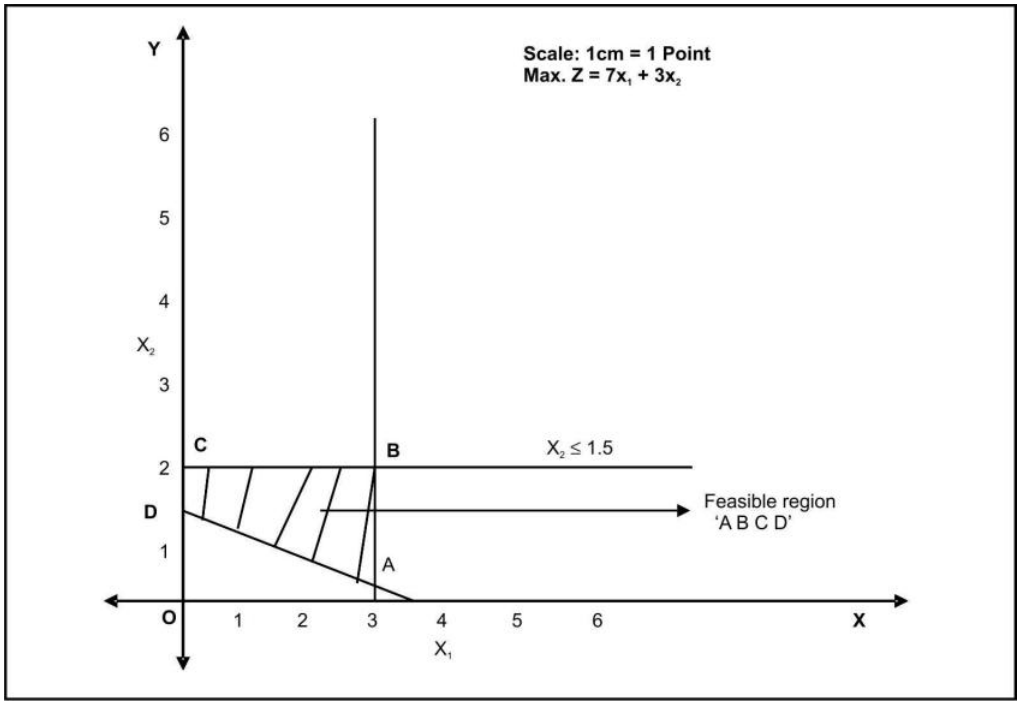
Step 2: Fix up the graphic scale

Maximum points = 4

Minimum points = 1

1 cm. = 1 point

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner Points	x_1	x_2
A	2.5	0.17
B	2.5	1.5
C	0	1.5
D	0	1

At $A: x_1 = 2.5$ …..(1)

$x_2 + 3x_2 = 3$ …..(2)

Put $x_1 = 2.5$ in eq. (2) ,

$2.5 + 3x_2 = 3$

$3x_2 = 3 - 2.5$

$3x_2 = 0.5$

$x_2 = 0.166$

Therefore $x_2 = 0.17$

Notes

Step 5: Substitute the co-ordinates of the corner points to the objective function.

Maximise $'Z' = 7x_1 + 3x_2$

At 'A', $Z = 7(2.5) + 3(0.17) = 18.01$

At 'B', $Z = 7(2.5) + 3(1.5) = 22$

At 'C', $Z = 7(0) + 3(1.5) = 4.5$

At 'D', $Z = 7(0) + 3(1) = 3$

Inference

Hence, the company can get maximum profit by producing 2.5 units of product A and 3 units of product B.

Example:

Maximise $'Z' = 5x_1 + 3x_2$ (Subject to constraints)

$x_1 + x_2 \leq 6$

$2x_1 + 3x_2 \leq 3$

$x_1 \leq 3$

$x_2 \leq 3$

$x_1, x_2 \geq 0$ (Non-negativity constraints)

Solution:

Step 1: Find the divisible of the equalities.

Equation	x_1	x_2
$x_1 + x_2 = 6$	6	6
$2x_1 + 3x_2 = 3$	1.5	1
$x_1 = 3$	3	0
$x_2 = 3$	0	3

Step 2: Fix up the graphic scale

Maximum points = 6

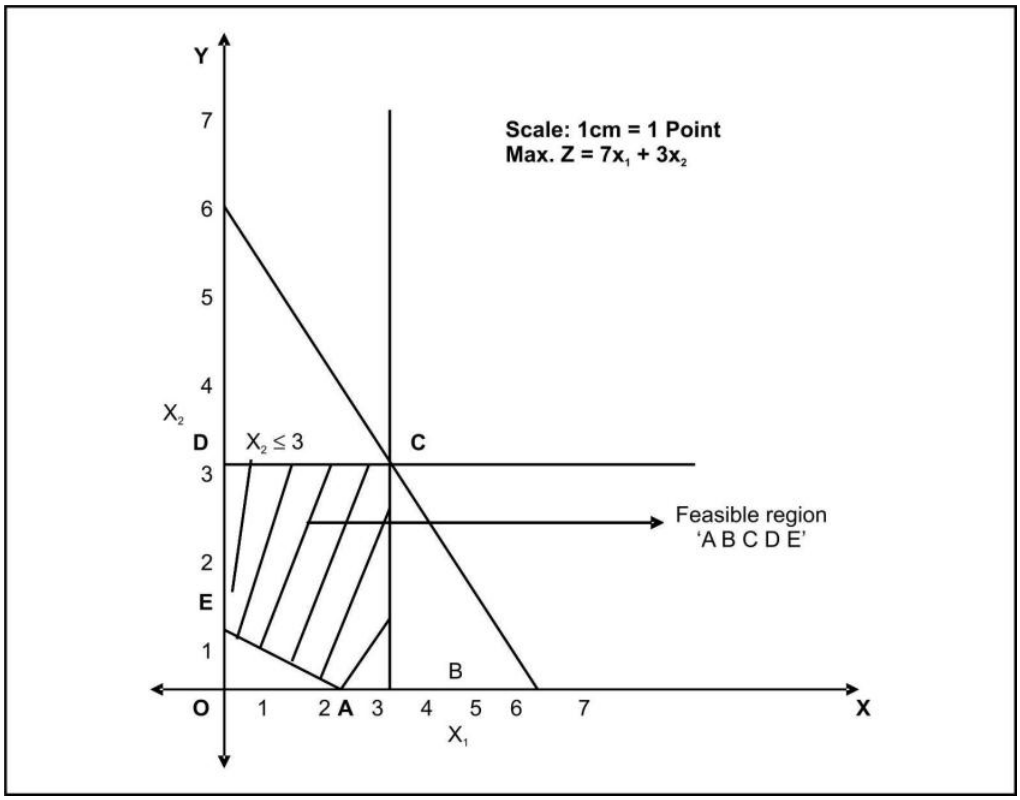
Minimum points = 1

1 cm. = 1 point

Step 3: Graph the data

Notes

Step 3: Graph the data



Step 4: Find the co-ordinates of the corner points

Corner Points	x_1	x_2
A	1.5	0
B	3	0
C	3	3
D	0	3
E	0	1

Step 5: Substitute the co-ordinate of the corner points to the objective function.

$$Z = 5x_1 + 3x_2$$

At 'A', Z = 5(1.5) + 3(0) = 7.5

At 'B', Z = 5(3) + 3(0) = 15

At 'C', Z = 5(3) + 3(3) = 24

At 'D', Z = 5(0) + 3(3) = 9

At 'E', Z = 5(0) + 3(1) = 3

Inference

Maximum profit (₹ 24) can be gained by producing 3 units of product 'M' and 3 units of product 'N'.

Notes

Case Study **Ace Air Lines**

The director of passenger services of Ace Air Lines was trying to decide how many new stewardesses to hire and train over the next six months. He had before him the requirements in number of stewardess flight hours needed.

Month	Hours needed
Jan	8,000
Feb	7,000
March	8,000
April	10,000
May	9,000
June	12,000

It took one month to train a stewardess before she was able to be used on regular flights. Hence, hiring had to be done a month before the need arose. Secondly, training of new stewardess required the time of already trained stewardess. It took approximately 100 hours of regular stewardess time for each trainee during the month of training period. In other words, the number of hours available for flight services by regular stewardesses was cut by 100 hours for each trainee.

The director of passenger services was not worried about January since he had 60 stewardesses available. Company rules required that a stewardess could not work more than 150 hours in any month. This meant that he had a maximum of 9,000 hours available for January, one thousand in excess of his need (stewardesses were not laid off in such cases merely worked fewer hours).

Company record showed that 10% of the stewardesses quit their jobs each month to be married or for other reasons.

The cost of Ace Air lines for a regular stewardesses was ₹ 800 per month for salary and fringe benefits, regardless of how many hours she worked. (She, of course, could not work more than 150 hours.) The cost of a trainee was ₹ 400 per month for salary and fringe benefits.

Question:

Formulate the above as a linear programming design to solve the problem of directory of passenger services at minimum cost. Be sure to identify all the symbols that you use and explain (briefly) all equations.

2.9 Summary

- Linear programming determines the way to achieve the best outcome (such as maximum profit or lowest cost) in a given mathematical model and some list of requirements represented as linear equations.
- It is a technique to ensure the optimum allocation of scarce resources in order to deliver for the fulfillment of ever increasing demands in the market.
- Linear Programming is used as a helping tool in nearly all functional areas of management.
- The graphical method to solve linear programming problem helps to visualize the procedure explicitly.
- It also helps to understand the different terminologies associated with the solution of LPP.

Notes

2.10 Keywords

Constraints: A condition that a solution to an optimization problem must satisfy.

Feasible Region: The region containing solution.

Feasible Solution: If a solution satisfies all the constraints, it is called feasible solution.

Shadow Price: The amount that the objective function value changes per unit change in the constraint.

2.11 Review Questions

1. Explain the linear programming problem giving two examples.
2. What are the essential characteristics of a linear programming model?
3. What do you understand by 'Graphical Method'? Give its limitations.
4. Explain the graphical method of solving a Linear programming Model involving two variables.
5. Define and explain the following:
   (i) Optimum Solution
   (ii) Feasible Solution
   (iii) Unrestricted Variables
6. A firm manufacturers headache pills in two sizes A and B. Size A contains I grains of aspirin, 5 grains of bicarbonate and 1 grain of codeine. It is found by uses that it requires at least 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codeine for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief.

   Formulate the problem as a standard LPP.
7. Consider a small plant which makes 2 types of automobile parts say A and B. It buys castings that are machined, bored and polished. The capacity of machining is 25 per hour for A and 40 hours for B, capacity of boring is 28 per hours for A and 35 per hour for B, and the capacity of polishing is 35 per hour A and 25 hour of B. Casting for port A costs ₹ 2 each and for part B they cost ₹ 3 each. They sell for ₹ 5 and ₹ 6 respectively. The three machines have running costs of ₹ 20, ₹ 14 and ₹ 17.50 per hour.

   Assuming that any combination of parts A and B can be sold, what product mix maximizes profit?
8. A ship is to carry 3 types of liquid cargo – X, Y and Z. There are 3,000 litres of X available, 2,000 litres of Y available and 1,500 litres of Z available. Each litre of X, Y and Z sold fetches a profit of ₹ 30, ₹ 35 and ₹ 40 respectively. The ship has 3 cargo holds-A, B and C of capacities 2,000, 2,500 and 3,000 litres respectively. From stability considerations, it is required that each hold be filled in the some proportion. Formulate the problem of loading the ship as a linear programming problem. State clearly what are the decision variables and constraints.
9. A company produces two types of pens, say A & B. Pen A is superior in quality while pen B is of lower quality. Net profits on pen A and B are ₹ 5 and ₹ 3 respectively. Raw material required for pen A is twice as that of pen B. The supply of raw material is sufficient only for

Notes

1,000 pens of B per day. Pen A requires a special nib and only 400 such nibs are available in a day. For pen B, only 700 nibs are available in a day.

Using graphical method, find the daily product mix so that the company can make maximum profits.

10. G.L Breweries Ltd. has two bottling plants located at Pune and Bangalore. Each plant produces three drinks; whisky, beer and brandy. The number of bottles produced in a day are as follows – whisky 1,500, beer 3,000 and brandy 2,000 at Pune and Whisky 1,500 beer 1,000 and brandy 5,000 at Bangalore. A market survey indicates that during November, there will be demand for 20,000 bottles of whisky, 40,000 bottles of beer and 44,000 bottles of brandy. The operating cost per day for plants at Pune and Bangalore are ₹ 600 and ₹ 400 respectively. For how many days each plant be run in November so as to meet the demand at minimum cost?

Answers: Self Assessment

1.	(d)	2.	(d)
3.	(d)	4.	Two
5.	Feasible region	6.	far most

2.12 Further Readings

Books

J.K. Sharma, *Operations Research, Theory and Applications,* MacMillan India Ltd.

Kanti Swarup, P.K Gupta & Manmohan, *Operations Research*, Sultan Chand Publications, New Delhi

Michael W. Carter, Camille C. Price, *Operations Research: A Practical Introduction,* CRC Press, 2001

Paul A. Jensen, Jonathan F. Bard, *Operations Research Models and Methods,* John Wiley and Sons, 2003

Richard Bronson, Govindasami Naadimuthu, *Schaum's Outline of Theory and Problems of Operations Research,* McGraw-Hill Professional; 1997

Online links

http://www.zweigmedia.com/RealWorld/simplex.html

www.math.ucla.edu/

www.math.ncsu.edu

Unit 3: Linear Programming Problem – Simplex Method

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Objectives

After studying this unit, you will be able to:

- Understand the meaning of word 'simplex' and logic of using simplex method
- Know how to convert a LPP into its standard form by adding slack, surplus and artificial variables
- Learn how to solve the LPP with the help of Big M methodology
- Understand the significance of duality concepts in LPP and ways to solve duality problems

Introduction

In practice, most problems contain more than two variables and are consequently too large to be tackled by conventional means. Therefore, an algebraic technique is used to solve large problems using Simplex Method. This method is carried out through iterative process systematically step by step, and finally the maximum or minimum values of the objective function are attained.

The simplex method solves the linear programming problem in iterations to improve the value of the objective function. The simplex approach not only yields the optimal solution but also other valuable information to perform economic and 'what if' analysis.

Notes

3.1 Simplex Method of Linear Programming

Under 'Graphical solutions' to LP, the objective function obviously should have not more than two decision variables. If the decision variables are more than two, the 'Cartesian Plane' cannot accommodate them. And hence, a most popular and widely used analysis called 'SIMPLEX METHOD', is used. This method of analysis was developed by one American Mathematician by name George B. Dantzig, during 1947.

This method provides an algorithm (a procedure which is iterative) which is based on fundamental theorems of Linear Programming. It helps in moving from one basic feasible solution to another in a prescribed manner such that the value of the objective function is improved. This procedure of jumping from one vertex to another vertex is repeated.

Steps:

1. Convert the inequalities into equalities by adding slack variables, surplus variables or artificial variables, as the case may be.
2. Identify the coefficient of equalities and put them into a matrix form AX = B

   Where "A" represents a matrix of coefficient, "X" represents a vector of unknown quantities and B represents a vector of constants, leads to AX = B [This is according to system of equations].
3. Tabulate the data into the first iteration of Simplex Method.

Table 3.1: Specimen

Basic (BV) Variable	C_B	X_B	Y_1	Y_2	S_1	S_2	Minimum Ratio $X_{Bi/Yij}$; $Y_{ij} > 0$
S_1 S_2							
		Z_j C_j					
		$Z_j - C_j$					

   (a) Cj is the coefficient of unknown quantities in the objective function.

   $Zj = \Sigma C_{BiYij}$ (Multiples and additions of coefficients in the table, i.e., $C_{B1} \times Y_{11} + C_{B2} \times Y_{12}$)

   (b) Identify the Key or Pivotal column with the minimum element of Zj - Cj denoted as 'KC' throughout to the problems in the chapter.

   (c) Find the 'Minimum Ratio' i.e., X_{Bi}/Y_{ij}.

   (d) Identify the key row with the minimum element in a minimum ratio column. Key row is denoted as 'KP'.

   (e) Identify the key element at the intersecting point of key column and key row, which is put into a box ☐ throughout to the problems in the chapter.
4. Reinstate the entries to the next iteration of the simplex method.

   (a) The pivotal or key row is to be adjusted by making the key element as '1' and dividing the other elements in the row by the same number.

   (b) The key column must be adjusted such that the other elements other than key elements should be made zero.

Notes

(c) The same multiple should be used to other elements in the row to adjust the rest of the elements. But, the adjusted key row elements should be used for deducting out of the earlier iteration row.

(d) The same iteration is continued until the values of $Z_j - C_j$ become either '0' or positive.

5. Find the 'Z' value given by C_B, X_B.

3.1.1 Maximisation Cases

Example:

Maximise $'Z' = 5x_1 + 3x_2$ [Subject to constraints]

$$x_1 + x_2 \leq 2$$
$$5x_1 + 2x_2 \leq 10$$
$$3x_1 + 8x_2 \leq 12$$

Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

Solution:

Step 1: Conversion of inequalities into equalities adding slack variables

$$x_1 + x_2 + x_3 = 2$$
$$5x_1 + 2x_2 + x_4 = 10$$
$$3x_1 + 8x_2 + x_5 = 12$$

Where, x_3, x_4 and x_5 are slack variables.

Step 2: Fit the data into the matrix form AX = B

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 & S_3 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 2 & 0 & 1 & 0 \\ 3 & 8 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = B = \begin{pmatrix} 2 \\ 10 \\ 12 \end{pmatrix}$$

Step 3: Fit the data into first iteration of Simplex Method

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	2	$\boxed{1}$	1	1	0	0	2/1 = 2(KR) →
S_2	0	10	5	2	0	1	0	10/5 = 2
S_3	0	12	3	8	0	0	1	12/3 = 4
		Z_j	0	0				
		C_j	5	3				
		$Z_j - C_j$	-5	-3				

(↑ KC)

Therefore, $Z = C_B X_B$

$= (0\times2) + (0\times10) + (0\times12)$

$= 0$

Notes

Step 4: Fit the data into second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
Y_1	5	2/1 = 2	1/1 = 1	1/1 = 1	–	–	–	–
S_2	0	10 – 2(5) = 0	5 – 1(5) = 0	2–1(5) = –3	–	–	–	–
S_3	0	12 – 2(3) = 6	3 – 1(3) = 0	8–1(3) = 5	–	–	–	–
		Z_j C_j	5 5	5 3				
		$Z_j - C_j$	0	2				

Therefore, $Z = C_B X_B$

$= (5 \times 2) + (0 \times 0) + (0 \times 6)$

Therefore, $Z = 10$

Therefore, Maximum value of 'Z' = 10

Example:

Maximise 'Z' = $2x_1 + 3x_2$ [Subject to constraints]

$x_1 + x_2 \leq 1$

$3x_1 + x_2 \leq 4$

Where, $x_1, x_2 \geq 0$

Solution:

Step 1: Conversion of inequalities into equalities by adding slack variables.

$x_1 + x_2 + x_3 = 1$

$3x_1 + x_2 + x_4 = 4$

Where x_3 and x_4 are slack variables.

Step 2: Identify the coefficients.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = B = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Step 3: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	Min. Ratio
S_1	0	1	1	$\boxed{1}$	1	0	1/1 = 1 (KR)?
S_2	0	4	3	1	0	1	4/1 = 4
		Z_j C_j	0 2	0 3			
		$Z_j - C_j$	-2	-3			

(↑ KC)

Notes

Therefore, $Z = C_B X_B$

$= 0 + 0 + 0$

$= 0$

Step 4: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	Min. Ratio
y_2	3	1/1 = 1	1/1 = 1	1/1 = 1	--	--	--
S_2	0	4-1(1) = 3	3-1(1) = 2	1-1(1) = 0	--	--	--
		Z_j	3	3			
		C_j	2	3			
		$Z_j - C_j$	1	0			

Therefore, $Z = C_B X_B$

$= (3 \times 1) + (0 \times 3)$

$= 3$

Therefore, Maximum value of 'Z' = 3

Example:

Maximise 'Z' = $4x_1 + 3x_2$ [Subject to constraints]

$2x_1 + x_2 \leq 30$

$x_1 + x_2 \leq 24$

Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

Solution:

Step 1: Convert the inequalities into equalities adding slack variables.

$2x_1 + x_2 + x_3 = 30$

$x_1 + x_2 + x_4 = 24$

Where x_3 and x_4 are slack variables.

Step 2: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 \\ x_1 & x_2 & x_3 & x_4 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = B = \begin{pmatrix} 30 \\ 24 \end{pmatrix}$$

Step 3: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	Min. Ratio
S_1	0	30	$\boxed{2}$		1	0	30/2 = 15 (KR) ?
S_2	0	24	1	1	0	1	24/1 = 24
		Z_j	0	0			
		C_j	4	3			
		$Z_j - C_j$	-4	-3			

(↑ KC)

Notes

Therefore, $Z = (0 \times 30) + (0 \times 24)$

$= 0$

Step 4: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	Min. Ratio
Y_1	4	30/2 = 15	2/2 = 1	½ = 0.5	-	-	15/0.5 = 30
S_2	0	24 – 15(1) = 9	1 – 1(1) = 2	1 – 0.5(1) = $\boxed{0.5}$	-	-	9/0.5 = 18 (KR)?
		Z_j C_j	4 4	2 3			
		$Z_j - C_j$	0	-1			

(↑ KC)

Therefore, $Z = C_B X_B$

$= (4 \times 15) + (0 \times 9)$

$= 60$

Step 5: Third iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	Min. Ratio
Y_1	4	15 – 18(0.5) = 6	1 – 0(0.5) = 1	0.5 – 1(0.5) = 0	-	-	-
S_2	3	9/0.5 = 18	0/0.5 = 0	0.5/0.5 = 1	-	-	-
		Z_j C_j	4 4	3 3			
		$Z_j - C_j$	0	0			

Therefore, $Z = C_B X_B$

$= (4 \times 6) + (3 \times 18)$

$Z = 78$

Example:

Maximise 'Z' = $5x_1 + 3x_2$ [Subject to constraints]

$x_1 + x_2 \leq 12$

$5x_1 + 2x_2 \leq 10$

$3x_1 + 8x_2 \leq 12$

Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

Solution:

Step 1: Convert the inequalities into equalities adding slack variables.

$x_1 + x_2 + x_3 = 12$

$5x_1 + 2x_2 + x_4 = 10$

$3x_1 + 8x_2 + x_5 = 12$

Where x_3, x_4 and x_5 are slack variables.

Notes

Step 2: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 & S_3 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 2 & 0 & 1 & 0 \\ 3 & 8 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad B = \begin{pmatrix} 12 \\ 10 \\ 12 \end{pmatrix}$$

Step 3: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	12	1	1	1	0	0	12/1 = 12
S_2	0	10	$\boxed{5}$	2	0	1	0	10/5 = 2(KR)?
S_3	0	12	3	8	0	0	1	12/3 = 4
		Z_j	0	0				
		C_j	5	3				
		$Z_j - C_j$	–5	–3				

(↑ KC)

Therefore, $Z = C_B X_B$

$= (0 \times 12) + (0 \times 10) + (0 \times 12) = 0$

Step 4: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	12 –2(1) = 10	1–1(1) = 0	1 – 0.4(1) = 0.6	–	–	–	10/0.6 = 16.67
y_1	5	10/5 = 2	5/5 = 1	2/5 = 0.4	–	–	–	–2/0.4 = 5
S_3	0	12 – 2(3) = 6	3 – 1(3) = 0	8 – 0.4(3) = 6.8	–	–	–	6/6.8 = 0.88(KR) ?
		Z_j	5	2				
		C_j	5	3				
		$Z_j - C_j$	0	–1				

(↑ KC)

Therefore, $Z = C_B X_B$

$= (0 \times 10) + (5 \times 2) + (0 \times 6) = 10$

Step 5: Third iteration of Simplex Method.

BV	C_B	X_B	y_1	y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	10 –0.88 (0.6) = 9.47	0	0.6 –1 (0.6) = 0	-	-	-	-
y_1	5	2 –0.88 (0.4) = 1.698	1	0.4 –1 (0.4) = 0	-	-	-	-
y_2	3	6/6.8 = 0.88	0	6.8/6.8 = 1	-	-	-	-
		Z_j	5	3				
		C_j	5	3				
		$Z_j - C_j$	0	0				

Therefore, $Z = C_B X_B$

$= (0 \times 9.47) + (5 \times 1.698) + (3 \times 0.88)$

Therefore, maximum value of Z = 10.88

Notes

Example:

Maximise $'Z' = 5x_1 + 3x_2$ [Subject to constraints]

$= 3x_1 + 5x_2 + \leq 15$

$= 5x_1 + 2x_2 \leq 10$

Where, $x_1, x_2 \geq 0$

Solution:

Step 1: Convert the inequalities into equalities by adding the slack variables.

$3x_1 + 5x_2 + x_3 = 15$

$5x_1 + 2x_2 + x_4 = 10$

Where, x_3 and x_4 are slack variables.

Step 2: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 \\ x_1 & x_2 & x_3 & x_4 \\ 3 & 5 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad B = \begin{pmatrix} 15 \\ 10 \end{pmatrix}$$

Step 3: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	Min. Ratio
S_1	0	15	3	5	1	0	15/3 = 5
S_2	0	10	$\boxed{5}$	2	0	1	10/5 = 2(KR) ?
		Z_j	0	0			
		C_j	5	3			
		$Z_j - C_j$	-5	-3			

(↑ KC)

Therefore, Maximise $Z = C_B X_B$

$= (0 \times 15) + (0 \times 10)$

$= 0$

Step 4: Second iteration of Simplex Method.

BV	C_B	X_B	y_1	Y_2	S_1	S_2	Min. Ratio
S_1	0	15 – 2 (3) = 9	3 –1(3) = 0	5 – 0.4 (3) = $\boxed{3.8}$	--	--	9/3.8 = 2.37 (KR)?
S_2	5	10/5 = 2	5/5 = 1	2/5=0.4	--	--	2/0.4 = 5
		Z_j	5	2			
		C_j	5	3			
		$Z_j - C_j$	0	–1			

(↑ KC)

Therefore, $Z = C_B X_B$

$= (0 \times 9) + (5 \times 2)$

$= 10$

Notes

Step 5: Third iteration of Simplex Method.

BV	C_B	X_B	y_1	y_2	S_1	S_2	Min. Ratio
y_2	3	9/3.8 = 2.37	0	3.8/3.8 = 1	-	-	-
y_1	5	2 –2.37(0.4) = 1.052	1	0.4 –1(0.4) = 0	-	-	-
		Z_j C_j	5 5	3 3			
		$Z_j - C_j$	0	0			

Therefore, Maximise Z $= C_B X_B$

$= (3 \times 2.37) + (5 \times 1.052)$

Maximum value of'Z' = 12.37

Notes Real life complex applications usually involve hundreds of constraints and thousands of variables. So, virtually these problems cannot be solved manually. For solving such problems, you will have to rely on employing an electronic computer.

Self Assessment

1. Solve the following LPP problem using simplex method.

   Maximize 'Z' $= 7x_1 + 5x_2$ [Subject to constraints]

   $x_1 + x_2 \leq 6$

   $4x_1 + 3x_2 \leq 12$

   Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

2. Solve the following LPP problem using simplex method.

   Maximise 'Z' $= 5x_1 + 7x_2$ [Subject to constraints]

   $= x_1 + x_2 \leq 4$

   $= 3x_1 - 8x_2 \leq 24$

   $= 10x_1 + 7x_2 \leq 35$

   Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

3.1.2 Minimization Cases

Example:

Minimize 'Z' $= -x_1 - 2x_2$ [Subject to constraints]

$-x_1 + 3x_2 \leq 10$

$x_1 + x_2 \leq 6$

$x_1 - x_2 \leq 2$

Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

Notes

Solution:

Step 1: Convert the minimization problem into maximisation case by changing the signs of the decision variables in the objective function.

Therefore, 'Z' = $x_1 + 2x_2$ [Subject to constraints]

Step 2: Convert the inequalities into equalities by adding slack variables.

$$-x_1 + 3x_2 + x_3 = 10$$

$$x_1 + x_2 + x_4 = 6$$

$$x_1 - x_2 + x_5 = 2$$

Where x_3, x_4 and x_5 are slack variables.

Step 3: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 & S_3 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ -1 & 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = B = \begin{pmatrix} 10 \\ 6 \\ 2 \end{pmatrix}$$

Step 4: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	10	–1	3	1	0	0	10/3 = 3.3 (KR)?
S_2	0	6	1	1	0	1	0	6/1 = 6
S_3	0	2	1	–1	0	0	1	–½ = – 0.5
		Z_j C_j	0 1	0 2				
		$Z_j - C_j$	–1	–2				

(↑ KC)

Therefore, $Z = C_B X_B$

$0 + 0 + 0 = 0$

Step 5: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
y_2	2	10/3 = 3.33	–1/3 = 0.33	3/3 = 1				3.33/–0.33 = –10.09
S_2	0	6 – 3.33 (1) 2.67	1 – 1(–0.33) (1) = 2.67	1–1(1) = 0 = 1.33				2.67/1.33 = 2.00 (KR) →
S_3	0	2 – 3.33 (–1) = 5.33	1–(–0.33) (–1) = 0.67	–1–1 (–1) = 0				5.33/0.67 = 8.00
		Z_j C_j	–0.66 1	2 2				
		$Z_j - C_j$	–1.66	0				

(↑ KC)

Therefore, $Z = C_B X_B$

$= (2 \times 3.33) + (2 \times 2.67) + (0 \times 5.33)$

$= 6.66 + 0 + 0 = 6.66$

Notes

Step 6: Third iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
y_2	2	3.33 – 2(–0.33) = 3.99	–0.33 – 1(–0.33) = 0	1	–	–	–	–
y_1	1	2.67/1.35 = 2	1.33/1.33 = 1	0	–	–	–	–
S_3	0	5.33 – 2(0.67) = 3.99	0.67 – 1(0.67) = 0	0	–	–	–	–
		Z_j C_j	1 1	2 2	–			
		$Z_j - C_j$	0	0	–	–		

Max. $Z = C_B X_B$

$= (2 \times 3.99) + (1 \times 2) + (0 \times 3.99)$

$= 7.98 + 2 + 0$

$= .98$

Therefore, min. $Z = -9.98$

Example:

Minimise 'Z' = $-x_1 - 3x_2 + 2x_3$ [Subject to constraints]

$3x_1 - x_2 + 3x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

Where, $x_1, x_2, x_3 \geq 0$ [Non-negativity constraints]

Solution:

Step 1: Conversion of the minimization case into maximisation case.

Therefore, Maximise $Z = -x_1 + 3x_2 - 2x_2$ [Subject to constraints]

Step 2: Convert of the inequalities into equalities by adding slack variables.

Therefore, $3x_1 - x_2 + 3x_3 + x_4 = 7$

$-2x_1 + 4x_2 + x_5 = 12$

$-4x_1 + 3x_2 + 8x_3 + x_6 = 10$

Where, x_4, x_5 and x_6 are slack variables.

Step 3: Fit the data into matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & Y_3 & S_1 & S_2 & S_3 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x \\ 3 & -1 & 3 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = B = \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix}$$

Notes

Step 4: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	Min. Ratio
S_1	0	7	3	–1	3	1	0	0	7/–1 = –7
S_2	0	12	–2	4	0	0		0	12/4 = 3 (KR)→
S_3	0	10	–4	3	8	0	0	1	10/3 = 3.33
		Z_j C_j	0 –1	0 3	0 –2				
		$Z_j - C_j$	1	–3	2				

(↑ KC)

Therefore, $Z = C_B X_B$

$= 0 + 0 + 0$

$= 0$

Step 5: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	Min. Ratio
S_1	0	7–3 (–1) = 10	3 + 0.5 (–1) = 2.5	–1+1 (–1) = 0	3–0 (–1) = 3	–	–	–	10/2.5 = 4 (KR)→
Y_2	3	12/4 = 3	–2/4 = –0.5	4/4 = 1	0	–	–	–	3/–0.5 = -
S_3	0	10 – 3 (3) = 1	–4 + (–0.5) (3) = –5.5	3–1 (3) = 0	8–0 (3) = 8	–	–	–	1/2.5 = -
		Z_j C_j	–1.5 –1	3 –3	0 –2				
		$Z_j - C_j$	–0.5	–0	2				

(↑ KC)

Therefore, $Z = C_B X_B$

$= (0 \times 0) + (3 \times 3) + (0 \times 1)$

$= 0 + 9 + 0$

$= 9$

Step 6: Third iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	Min. Ratio
y_1	–1	10/2.5 = 4	2.5/2.5 = 1	0	3/2.5 = 1.2	–	–	–	–
S_2	3	3–4 (–0.5)= 5	–0.5–1 (–0.5)= 0	1	0	–	–	–	–
S_3	0	1–4 (–2.5)= 11	–2.5–1 (–2.5)= 0	0	8 – 1 (–2.5) = 10.5	–	–	–	–
		Zj Cj	–1 –1	3 3	–1.2 –2				
		Zj–Cj	0	0	0.8				

Therefore, Maximise ‘Z’ = $C_B X_B$

$= -4 + 15 + 0$

$= 11$

Therefore, Minimise $Z = -11$

Notes

Example:

Minimise $\quad$ 'Z' = $-10x_1 - 12x_2 - 15x_3$ $\quad$ [Subject to constraints]

$$0.10x_1 + 12x_2 + 0.15x_3 \le 36$$
$$0.06x_1 + 0.05x_2 + 0.09x_3 \le 30$$
$$0.18x_1 + x_2 + 0.07x_3 \le 37$$
$$0.13x_1 + 0.10x_2 + 0.08x_3 \le 38$$
$$x_1 \le 200$$
$$x_2 \le 100$$
$$x_3 \le 180$$

Where, $x_1, x_2, x_3 \ge 0$ $\quad$ [Non-negativity constraints]

Solution:

Step 1: Conversion of minimization case into maximisation case.

Therefore, Maximise 'Z' = $10x_1 + 12x_2 + 15x_3$ $\quad$ [Subject to constraints]

Step 2: Convert the inequalities into equalities adding slack variables.

$$0.10x_1 + 0.12x_2 + 0.15x_3 + x_4 = 36$$
$$0.06x_1 + 0.05x_2 + 0.09x_3 + x_5 = 30$$
$$0.18x_1 + x_2 + 0.07x_3 + x_6 = 37$$
$$0.13x_1 + 0.10x_2 + 0.08x_3 + x_7 = 38$$
$$x_1 + x_8 = 200$$
$$x_2 + x_9 = 100$$
$$x_3 + x_{10} = 180$$

Where, x_4, x_5, x_6, x_7, x_8, x_9 and x_{10} are slack variables.

Step 3: Fit the data into matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & Y_3 & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \\ 0.10 & 0.12 & 0.15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.06 & 0.05 & 0.09 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.18 & 1 & 0.07 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0.13 & 0.10 & 0.08 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{pmatrix} = B = \begin{pmatrix} 36 \\ 30 \\ 37 \\ 200 \\ 200 \\ 100 \\ 180 \end{pmatrix}$$

Notes

Step 4: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	S_4	S_5	S_6	S_7	Min. Ratio
S_1	0	36	0.10	0.12	0.15	1	0	0	0	0	0	0	36/0.15 = 240
S_2	0	30	0.06	0.05	0.09	0	1	0	0	0	0	0	30/0.09 = 33.3
S_3	0	37	0.18	1	0.07	0	0	1	0	0	0	0	37/0.08 = 529
S_4	0	38	0.13	0.10	0.08	0	0	0	1	0	0	0	38/0.08 = 475
S_5	0	200	1	0	0	0	0	0	0	1	0	0	200/0 = –
S_6	0	100	0	1	0	0	0	0	0	0	1	0	100/0 = –
S_7	0	180	0	0	$\boxed{1}$	0	0	0	0	0	0	1	180/1 = 180(KR)?
		Z_j C_j	0 10	0 12	0 15								
		$Z_j - C_j$	–10	–12	–15								

(↑ KC)

Therefore, $Z = C_B X_B$

$= (0 \times 36) + (0 \times 30) + (0 \times 37) + (0 \times 38) + (0 \times 200) + (0 \times 100) + (0 \times 180)$

$= 0 + 0 + 0 + 0 + 0 + 0 + 0$

$= 0$

Step 5: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	S_4	S_5	S_6	S_7	Min. Ratio
S_1	0	36 – 180(0.15) = 9	0.10–0(0.15) = 0.10	0.12–0 (0.15) = 0.12	0.15–0 (0.15) = 0	–	–	–	–	–	–	–	9/0.12 = 75
S_2	0	30 – 180(0.09) = 13.8	0.06–0 (0.09) = 0.06	0.05–0 (0.09) = 0.05	0.09 – 1 (0.09) = 0	–	–	–	–	–	–	–	13.80/0.05 = 276
S_3	0	37 – 180 (0.07)= 24.4	0.18–0 (0.07) = 0.18	1–0(0.07) = $\boxed{1}$	0.07–1 (0.07) = 0	–	–	–	–	–	–	–	24.4/1 = 24.4(KR)?
S_4	0	38 – 180 (0.08) = 23.6	0.13–0 (0.08) = 0.13	0.10 – 0 (0.08) = 0.10	0.08 – 1 (0.08) = 0	–	–	–	–	–	–	–	23.6/0.10 = 236
S_5	0	200–180	0.10 (0) = 1	0 – 0 = 0	0	–	–	–	–	–	–	–	(0) = 200
S_6	0	100–180 (0) = 100	0 – 0 = 0	1 – 0 (0)= 1	0	–	–	–	–	–	–	–	100/1= 100
Y_3	15	180/1 = 180	0	0	1/1 = 1	–	–	–	–	–	–	–	
		Z_j C_j	0 10	0 12	15 15								
		$Z_j - C_j$	–10	–12	0								

(↑ KC)

Therefore, $Z = C_B X_B$

$= (0 \times 9) + (0 \times 3.8) + (0 \times 24.4) + (0 \times 23.6) + (0 \times 200) + (0 \times 100) + (15 \times 180)$

$= 2700$

Notes

Step 6: Third iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	S_4	S_5	S_6	S_7	Min. Ratio
S_1	0	9 – 24.4 (0.12) = 6.07	0.10 – 0.18 (0.12= $\boxed{0.078}$	0.12 – 1 (0.12) = 0	0	–	–	–	–	–	–	–	6.072/0.078 = 77.85 (KR) →
S_2	0	13.8–24.4 (0.05)=12.58	0.06 – 18 (0.05) =0.05	0.05 – 1 (0.05) = 0	0	–	–	–	–	–	–	–	12.58/0.051 = 246.67
S_3	0	24.4/1 = 24.4	0.18/1 = 0.18	1/1 = 1	0	–	–	–	–	–	–	–	24.4/0.18 = 135.56
S_4	0	23.6–24.4 (0.10)=21.16	0.13 – 0.18 (0.10) = 0.112	0.10 – 1 (0.10) = 0	0	–	–	–	–	–	–	–	21.16/0.112 = 188.93
S_5	0	200 – 24.4 (0) = 200	1–0.18 (0)=1	0	0	–	–	–	–	–	–	–	200/1 = 200
S_6	0	100 – 24.4 (1) = 75.6	0 – 0.18 (1) = –0.18	1 – 1 (1) =0	0	–	–	–	–	–	–	–	75.6/–0.18 = –420
Y_3	15	180 – 0 =180	0	0	1	–	–	–	–	–	–	–	180/0 = 0
		Z_j C_j	2.16 10	12 12	15 15								
		$Z_j - C_j$	–7.84 (↑ KC)	0	0								

Therefore, $Z = C_B X_B$

$= (0\times6.072) + (0\times12.58) + (12\times24.4) + (0\times21.16) + (0\times200) + (0\times75.6) + (15\times180)$

$= 2,992.8$

Step 7: Fourth iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	S_4	S_5	S_6	S_7	Min. Ratio
y1	10	6.072/0.078 = 77.85	0.078/0.078 = 1	0	0	–	–	–	–	–	–	–	–
S2	0	12.58 – 77.85(0.051)= 8.61	0.051 – 1(0.051) = 0	0	0	–	–	–	–	–	–	–	–
y2	12	24.4 – 77.85(0.18)= 10.387	(0.18) – 1(0.18) = 0	1	0	–	–	–	–	–	–	–	–
S4	0	21.16 – 77.85(0.112)= 12.44	0.112 – 1(0.112) = 0	0	0	–	–	–	–	–	–	–	–
S5	0	200 – 77.85(1)= 122.15	1 – 1(1) = 0	0	0	–	–	–	–	–	–	–	–
S6	0	75.6 – 77.85(–0.18)= 89.613	–0.18 – 1(–0.18) = 0	0	0	–	–	–	–	–	–	–	–
y3	15	180 – 77.85(0) = 180	0	0	1	–	–	–	–	–	–	–	–
		Z_j C_j	10 10	12 12	15 15								
		$Z_j - C_j$	0	0	0								

Notes

Maximise $Z = C_B X_B$

$= (10 \times 77.85) + (12 \times 10.387) + (15 \times 180) + 0 + 0 + 0 + 0$

$= 778.5 + 124.644 + 2{,}700 + 0 + 0 + 0 + 0$

$= 3{,}603.144$

Therefore, Minimise Z = –3,603.144

Steps to Sum-up

1. Prepare the Table No. 1 and find the cost coefficients (Z_j) for different columns of A, i.e., for Y_i, by multiplying C_B column with entries of Y_j and adding the products, i.e., Z_j - $\Sigma C_{Bi} Y_{ij}$ and C_j is the most of coefficients for Z_j in objective function. Then, find the difference of Z_j and C_j ($Z_j - C_j$) for different columns of A.
   (a) *Incoming Vector:* The jth vector, i.e., y_j enters the basis if $Z_j - C_j$ is minimum where y_j is the jth column of the coefficient matrix 'A'.
   (b) *Outgoing Vector:* Find the ratio of XB_i/y_{ij} (for all $Y_{ij} > 0$) for all the elements of the incoming vector. Then, the vector attached to the row having minimum ratio would be removed from the basis (if y_{ij} is greater than '0', otherwise that item should be neglected).
   (c) *Key Element or Pivotal Element:* The element common to the incoming and outgoing vector is called key element or pivotal element.
2. The incoming vector has the coefficient of objective function in CB. Hence, make the pivotal element as "1" by dividing that row completely by the pivotal element. The other elements of the incoming vector other than pivotal element must be made "0"/"Zero". This can be done by deducting the elements of the respective rows by "K" times the adjusted pivotal row elements completely. The constant 'K' is chosen such that the pivotal columns element(s) is made "0"/"Zero". Then find Z_j and C_j in the usual manner of matrix method and if $Z_j - C_j$ is greater than or equal to zero for all columns, then the basic feasible solution is optimum otherwise the same procedure is to be continued.

Notes **Brief Steps of the simplex method:**

1. Convert the inequalities into equalities.
2. Identify the coefficients of equalities & put them into a matrix form.
3. Tabulate the data into 1st iteration of simplex method.
4. Reinstate entries in the 2nd iteration.
5. Find the 'Z' value.

Self Assessment

Fill in the blanks:

3. The element common to the incoming and outgoing vector is called …….. .
4. A ……. variable represents unused resources and are added to original objective function with zero coefficients.
5. A ……. Variable represents amount by which solution value exceed a resource.

Notes

3.2 Big 'M' Method

When the Linear Programming problem has greater than or equal to types of equations as constraints, it is obvious that some quantity should be deducted to convert them into equalities. The variables attached to it are known as 'surplus variables'. If the inequality is of the type greater than or equal to then add surplus variables which carry a negative sign and their cost coefficients in the objective function would be zero. As they would be considering slack or artificial variables initially as basic variables, and as the surplus variables carry negative signs, they represent vectors of identify matrix of the form,

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{etc.,}$$

which cannot be taken into the basis. Hence, artificial variables along with the surplus variables would be added.

These artificial variables carry large negative values (-m) in the objective function. The artificial variables can also be added to the equation. This helps in choosing the initial variable or variables for the basis. The slack variables then would go to the basis whose cost coefficients are supposed to be zero (0) and the cost coefficients of artificial variables are supposed to be -M for maximization cases and +M for Minimization cases. The procedure for iteration follows when Simplex technique to obtain the optimum solution is used. Since the method involves artificial variables carrying -M as the cost coefficient, where M is a very large number which helps in the optimum solution finding and hence it is known as 'Big M Method'.

Steps:

1. Express the problem in the standard form by using slack, surplus and artificial variables.
2. Select slack variables and artificial variables as the initial basic variables with the cost coefficients as '0' or '-M' respectively.
3. Use simplex procedure for iterations & obtain optimum solution. During the iterations, one can notice that the artificial variables leave the basis first and then the slack variables with improved value of objective function at each iteration to obtain the optimum solution.

Example:

Minimise 'Z' = $4x_1 + 8x_2 + 3x_3$ [Subject to constraints]

$x_1 + x_2 \geq 2$

$2x_1 + x_3 \geq 5$

Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

Solution:

Step 1: Conversion of minimization case into maximisation case.

Therefore, 'Z' = $-4x_1 - 8x_2 - 3x_3$ [Subject to constraints]

Step 2: Conversion of inequalities into equalities adding slack variables and artificial variables.

$-x_1 + x_2 - x_4 + x_6 = 2$

$2x_1 + x_3 - x_5 + x_7 = 5$

Where, x_4 and x_5 are surplus variables, x_6 and x_7 are artificial variables.

Notes

Step 3: Bring the objective function into a standard form.

Therefore, Maximise 'Z' = $-4x_1 - 8x_2 - 3x_3 + 0x_4 + 0x_5 - Mx_6 - Mx_7$

Step 4: Find out the matrix form of equalities.

$$A = \begin{pmatrix} Y_1 & Y_2 & Y_3 & S_1 & S_2 & a_1 & a_2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & -1 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Step 5: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	a_1	a_2	Min. Ratio
a_1	–M	2	$\boxed{1}$	$\boxed{1}$	0	–1	0	1	0	2/1 = 2(KR)→
a_2	–M	5	2	0	1	0	–1	0	1	5/2 = 2.5
		Z_j	–3M	–M	–M					
		C_j	–4	8	–3					
		$Z_j - C_j$	–3M+4	–M+8	–M+3					

(↑ KC)

Therefore, $Z = C_B X_B$

$= (-M \times 2) + (-M \times 5)$

$= -2M - 5M$

$= -7M$

Step 6: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	a_1	a_2	Min. Ratio
y_1	–4	2/1 = 2	1/1 = 1	1	0	–1	0	1	0	2/0 = –
a_2	–M	5 – 2(2) = 1	2 –1 (2) = –2	0–1 (2) = –2	1–0 (2) = 1	–	–	–	–	1/1 = 1 (KR)→
		Z_j	–4	–4 + 2M	–M					
		C_j	–4	–8	–3					
		$Z_j - C_j$	0	4 + 2M	M+3					

(↑ KC)

Therefore, $Z = (-4 \times 2) + (-M \times 1)$

$= -8 - M$

Step 7: Second iteration of Simplex Method.

BV	C_B	X_B	y_1	y_2	y_3	S_1	S_2	a_1	a_2	Min. Ratio
y_1	–4	2–1 (0) = 2	1–1 (0) = 1	1 – (–2) (0) = 1	0	–	–	–	–	–
y_3	–3	1/1 = 1	0	–2/1 = –2	1/1 = 1	–	–	–	–	–
		Z_j	–4	2	–3					
		C_j	–4	–8	–3					
		$Z_j - C_j$	0	10	0					

Notes

Therefore, Maximize $Z = C_B X_B$

$= (-4 \times 2) + (-3 \times 1)$

$= -8 - 3$

$= -11$

Therefore, Minimize $Z = 11$

Example:

Minimise $'Z' = 3x_1 + 5x_2$ [Subject to constraints]

$2x_1 + 8x_2 \geq 40$

$3x_1 + 4x_2 \geq 50$

$x_1, x_2 \geq 0$ [Non-negativity constraints]

Solution:

Step 1: Convert the above minimization case into maximisation case.

Therefore, Maximise $Z = -3x_1 - 5x_2$

Step 2: Convert the inequalities into equalities adding slack variables and artificial variables.

$2x_1 + 8x_2 - x_3 + x_5 = 40$

$3x_1 + 4x_2 - x_4 + x_6 = 50$

Where, x_3 & x_4 are surplus variables and x_5 and x_6 are artificial variables.

Step 3: Bring the objective function into a standard form.

Maximise $'Z' = -3x_1 - 5x_2 - 0x_3 + 0x_4 - Mx_5 - Mx_6$

Step 4: Fit the data into a matrix form.

$$A = \begin{pmatrix} y_1 & y_2 & S_1 & S_2 & a_1 & a_2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 2 & 8 & -1 & 0 & 1 & 0 \\ 3 & 4 & 0 & -1 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = B = \begin{pmatrix} 40 \\ 50 \end{pmatrix}$$

Step 5: First iteration of Simplex Method.

BV	C_B	X_B	y_1	y_2	S_1	S_2	a_1	a_2	Min. Ratio
a_1	–M	40	2	8	–1	0	1	0	40/8 = 5 (KR)→
a_2	–M	50	3	4	0	–1	0	1	50/4 = 12.5
		Z_j C_j	–5M –3	–12M –5					
		$Z_j - C_j$	–5M + 3	12M + 5					

Therefore, $Z = C_B X_B$

$= (40x - M) + (50x - M)$

Notes

$= -40M - 50M$

$= -90M$

Step 6: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	a_1	a_2	Min. Ratio
y_2	–5	40/8 = 5	2/8 = 0.25	8/8 = 1	–	–	–	–	5/0.25 = 20
a_2	–M	50–5(4) = 30	3–0.25(4) = $\boxed{2}$	4–1(4)= 0	–	–	–	–	30/2 = 15(KR)→
		Z_j C_j	–2M – 1.25 –3	–5 –5					
		$Z_j - C_j$	–2M + 1.75	0					

(↑ KC)

Therefore, $Z = (-5 \times 5) + (-M \times 30)$

$= -30M - 25$

Step 7: Third iteration of Simplex Method.

7	C_B	X_B	Y_1	Y_2	S_1	S_2	a_1	a_2	Min. Ratio
y_2	–5	5–15 (0.25) = 1.25	0.25 – 1 (0.25) = 0	1–0 (0.25) = 1	–	–	–	–	–
y_1	–3	30/2 = 15	2/2 = 1	0/2 = 0	–	–	–	–	–
		Z_j C_j	–3 –3	–5 –5					
		$Z_j - C_j$	0	0					

$Z = C_B X_B$

$= -6.25 - 45$

Maximise $Z = -51.25$

Therefore, Minimise $Z = 51.25$

Example:

Minimise 'Z' = $12x_1 + 20x_2$ [Subject to constraints]

$6x_1 + 8x_2 \geq 100$

$7x_1 + 12x_2 \geq 120$

Where, $x_1, x_2 \geq 0$

Solution:

Step 1: Conversion of the minimization case into maximisation case.

Maximise $Z = -12x_1 - 20x_2$ [Subject to constraints]

Step 2: Convert the inequalities into equalities adding artificial and surplus variables.

$6x_1 + 8x_2 - x_3 + x_5 = 100$

$7x_1 + 12x_2 - x_4 + x_6 = 120$

Where, x_3 and x_4 are surplus variables and x_5 and x_6 are artificial variables.

Notes

Step 3: Bring the objective function into a standard form.

Maximise 'Z' = – $12x_1$ – $20x_2$ – x_3 + x_4 – Mx_5 – Mx_6

Step 4: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & S_1 & S_2 & a_1 & a_2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 6 & 8 & -1 & 0 & 1 & 0 \\ 7 & 12 & 0 & -1 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = B = \begin{pmatrix} 100 \\ 120 \end{pmatrix}$$

Step 5: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	a_1	a_2	Min. Ratio
a_1	–M	100	6	8	–1	0	1	0	100/8 = 12.5
A_2	–M	120	7	12	0	–1	0	1	120/12 = 10(KR)→
		Z_j C_j	–100M 12	–120M –20					
		$Z_j - C_j$	–100M + 12	120M+20					

(↑ KC)

Therefore, $Z = C_B X_B$

$= -100M - 120M$

$= -220M$

Step 6: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	a_1	a_2	Min. Ratio
a_1	–M	100–10 (8) = 20	6–0.5(8) = 1.33	8–1(8) = 0	–	–	–	–	20/1.33 = 15.04(KR)→
y_2	–20	120/12 = 10	7/12 = 0.58	12/12 = 1	–	–	–	–	10/0.58 = 17.24
		Z_j C_j	–1.33M–11.6–12	–20 –20					
		$Z_j - C_j$	–1.33M + 0.4	0					

(↑ KC)

Therefore, $Z = C_B X_B$

$= (-M \times 20) + (-20 \times 10)$

$= -20M - 200$

Step 7: Third iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	a_1	a_2	Min. Ratio
y_1	–12	20/1.33 = 15.04	1.33/1.33 = 1	0	–	–	–	–	––
y_2	–20	10 – 15.04 (90.58) = 1.28	0.58–1(0.58) = 0	1	–	–	–	–	––
		Zj Cj	–12 –12	–20 –20					
		Zj – Cj	0	0					

$Z = C_B X_B$

Notes

$= (-12 \times 15.04) + (-20 \times 1.28)$

$= -180.48 - 25.6$

Therefore, Maximise $Z = -206.08$

Therefore, Minimise $Z = 206.08$

Example: Solve the following LPP using Simplex Method.

Maximise 'Z' = $x_1 + 1.5x_2 + 2x_3 + 5x_4$ [Subject to constraints]

$$3x_1 + 2x_2 + 4x_3 + x_4 \le 6$$
$$2x_1 + x_2 + x_3 + 5x_4 \le 4$$
$$2x_1 + 6x_2 - 8x_3 + 4x_4 = 0$$
$$x_1 + 3x_2 - 4x_3 + 3x_4 = 0$$
$$\text{Where, } x_1, x_2, x_3, x_4 \ge 0$$

Solution:

Step 1: Convert the inequalities into equalities by adding slack variables and surplus variables.

$$3x_1 + 2x_2 + 4x_3 + x_4 + x_5 = 6$$
$$2x_1 + x_2 + x_3 + 5x_4 + x_6 = 4$$
$$2x_1 + 6x_2 - 8x_3 + 4x_4 + x_7 = 0$$
$$x_1 + 3x_2 - 4x_3 + 3x_4 + x_8 = 0$$

Where, x_5 and x_6 are slack variables and x_7 and x_8 are artificial variables.

Step 2: The standard form of objective function.

Maximise 'Z' = $x_1 + 1.5x_2 + 2x_3 + 5x_4 \pm 0x_5 \pm 0x_6 - Mx_7 - Mx_8$

Step 3: Fit the data into a matrix form.

$$A = \begin{pmatrix} Y_1 & Y_2 & Y_3 & y_4 & S_1 & S_2 & a_1 & a_2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ 3 & 2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 5 & 0 & 1 & 0 & 0 \\ 2 & 6 & -8 & 4 & 0 & 0 & 1 & 0 \\ 1 & 3 & -4 & 3 & 0 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = B = \begin{pmatrix} 6 \\ 4 \\ 0 \\ 0 \end{pmatrix}$$

Step 4: First iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	Y_4	S_1	S_2	a_1	a_2	Min. Ratio
S_1	0	6	3	2	4	1	1	0	0	0	6/2 = 3
S_2	0	4	2	1	1	5	0	1	0	0	4/1 = 4
a_1	–M	0	2	6	–8	4	0	0	1	0	0/6 = 0(KR)?
a_2	–M	0	1	3	–4	3	0	0	0	1	0/3 = 0
		Z_j	–3M	–9M	12M	–7M					
		C_j	1	1.5	2	5					
		$Z_j - C_j$	–3M–1	–9M–1.5	+12M–2	–7M–5					

(↑ KC)

Notes

Therefore, $Z = C_B X_B$

$= (0 \times 6) + (0 \times 4) + (-M \times 0) + (-M \times 0)$

$= 0$

Step 5: Second iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	Y_4	S_1	S_2	a_1	a_2	Min. Ratio
S_1	0	6–0 (2)=6 (2)	3–0.33=2.34	2–1 (2)=0 (2)	4–1 (–1.33)=6.66 (2)	1– (0.67)=–0.34	–	–	–	–	6/–0.34 = –
S_2	0	4–0 (1)=4 (1)	2–0.33=1.67	1–1 (1)=0 (1)	1– (–1.33)=2.33 (1)	5– (0.67)=0.92	–	–	–	–	4/4.33
y_2	1.5	0/6 = 0	2/6 = 0.33	6/6 = 1	–8/6 = –1.33	4/6 = 0.67	–	–	–	–	0/0.67 = 0
A_2	–M	0–0 (3)=0	1–0.33 (3)=0.01	3–1 (3)=0	–4– (–1.33)(3)= 0.01	3 – 0.67 (3)=0.99	–	–	–	–	0/0.99 = 0 (KR)?→
		Z_j	0.5–0.01M	1.5	–2+0.01M	1.01–0.99M					
		C_j		1.5							
		$Z_j - C_j$	–0.5–0.01M	0	–4+0.01M	–3.99–0.99M					

(↑ KC)

Therefore, $Z = C_B X_B$

$= 0 + 0 + 0 + 0$

$= 0$

Step 6: Third iteration of Simplex Method.

BV	C_B	X_B	Y_1	Y_2	Y_3	Y_4	S_1	S_2	a_1	a_2	Min. Ratio
S_1	0	6–0 (–0.34) = 6	2.34–0.01 (–0.34) = 2.34	0	–0.34–1(–0.34) = 0	6.66– (–1) = 6.67	–	–	–	–	6/6.67 = 0.9 (KR) →
S_2	0	4 – 0 (4.33) = 4 (4.33)= 1.63	1.67–0.01 (4.33) = 0	0	4.33–1(4.33) = 0	2.33– (–0.01)	–	–	–	–	4/2.37 = 1.69
y_2	1.5	0–0 (0.67)= 0 (0.67)= 0.32	0.33–0.01(0.67) = 0	1	0.67–1 (0.67) = –1.32	–1.33–(–0.01) = –1.32	–	–	–	–	0/–1.32 = –
y_4	5	0/0.99 = 0 9 = 0.01	0.01/0.9 = 1	0	0.99/0.99 = 0–0.01	–0.01/0.09	–	–	–	–	0/–0.01 = –
		Z_j	0.53	1.5	5	2.03					
		C_j	1	1.5	5	2					
		$Z_j - C_j$	0.47	0	0	–4.03					

(↑ KC)

Therefore, $Z = C_B X_B$

$= 0 \times 6 + 0 \times 4 + 1.5 \times 0 + 5 \times 0$

$= 0$

Step 7: Fourth iteration of Simplex Method

BV	C_B	X_B	Y_1	Y_2	Y_3	Y_4	S_1	S_2	a_1	a_2	Min. Ratio
y_3	2	6/6.67 = 0	2.34/6.67 = 0.35	0	6.67/6.67 = 1	0	–	–	–	–	–
S_2	0	4– (0.9) (2.37) = 1.87	1.63 – 0.35 (2.37) = 0.80	0	2.37–1 (2.37)= 0	0	–	–	–	–	–
y_2	1.5	0–0.9 (–1.32) = 1.19	0.32–0.35 (–1.32) = 0.78	1	–1.32 – 1 (–1.32) = 0	0	–	–	–	–	–
y_4	5	0–0.9 (–0.01) = 0.009	0.01–0.35 (–0.01) = 0.014	0	–0.01–1 (–0.01) = 0	1	–	–	–	–	–
		Z_j	3.27	1.5	2	5					
		C_j	1	1.5	2	5					
		$Z_j - C_j$	2.27	0	0	0					

Notes

Therefore, $'Z' = C_B X_B$

$= (2 \times 0.9) + (0 \times 1.87) + (1.5 \times 1.19) + (5 \times 0.009)$

$= 1.8 + 0 + 1.79 + 0.045$

$= 3.64$

Did u know? Big 'M' Method is also known as 'Charnes' 'M' Technique.

Caution If the objective function z is to be minimized, then a very large positive price (called penalty) is assigned to each artificial variable. Similarly, if Z is to be maximized, then a very large negative price (also called penalty) is assigned to each of these variables. The only visible difference between these two penalty is that the one will be designated by -M for a maximization problem and +M for a minimization problem, where M>0.

Self Assessment

6. Solve the following LPP using the Big M method.

   Maximise $'Z' = 40x_1 + 60x_2$ [Subject to constraints]

   $2x_1 + x_2 \leq 70$

   $x_1 + x_2 \geq 40$

   $x_1 + x_2 \geq 40$

   $x_1 + 3x_2 \leq 90$

   Where, $x_1, x_2 \geq 0$

7. Solve the following LPP using the Big M method.

   Maximise $'Z' = 5x_1 + 3x_2$ [Subject to constraints]

   $x_1 + x_2 \leq 6$

   $2x_1 + 3x_2 \geq 3$

   $x_1 \leq 3$

   $x_2 \leq 3$

   Where, $x_1, x_2 \geq 0$

3.3 Unconstrained Variables

Sensitivity analysis involves 'what if?' questions. In the real world, the situation is constantly changing like change in raw material prices, decrease in machinery availability, increase in profit on one product, and so on. It is important to decision makers for find out how these changes affect the optimal solution. Sensitivity analysis can be used to provide information and to determine solution with these changes.

Sensitivity analysis deals with making individual changes in the coefficient of the objective function and the right hand sides of the constraints. It is the study of how changes in the coefficient of a linear programming problem affect the optimal solution.

Notes

We can answer questions such as,

1. How will a change in an objective function coefficient affect the optimal solution?
2. How will a change in a right-hand side value for a constraint affect the optimal solution?

For example, a company produces two products x_1 and x_2 with the use of three different materials 1, 2 and 3. The availability of materials 1, 2 and 3 are 175, 50 and 150 respectively. The profit for selling per unit of product x_1 is ₹ 40 and that of x_2 is ₹ 30. The raw material requirements for the products are shown by equations, as given below.

$$Z_{max} = 40x_1 + 30x_2$$

Subject to constraints

$$4x_1 + 5x_2 \leq 175 \quad \text{(a)}$$

$$2x_2 \leq 50 \quad \text{(b)}$$

$$6x_1 + 3x_2 \leq 150 \quad \text{(c)}$$

where $x_1, x_2 \geq 0$

The optimal solution is

x_1 = ₹ 12.50

x_2 = ₹ 25.00

Z_{max} = 40 × 12.50 + 30 × 25.00

= ₹ 1250.00

The problem is solved using TORA software and the output screen showing sensitivity analysis is given in Figure 3.1.

Figure 3.1: Sensitivity Analysis Table Output

Notes

3.3.1 Change in Objective Function Coefficients and Effect on Optimal Solution

Referring to the current objective coefficient of Figure 2.4, if the values of the objective function coefficient decrease by 16 (Min. obj. coefficient) and increase by 20 (Max. obj. coefficient) there will not be any change in the optimal values of $x_1 = 12.50$ and $x_2 = 25.00$. But there will be a change in the optimal solution, i.e. Z_{max}.

Notes This applies only when there is a change in any one of the coefficients of variables i.e., x_1 or x_2. Simultaneous changes in values of the coefficients need to apply for 100 Percent Rule for objective function coefficients.

For x_1,	Allowable decrease	=	Current value – Min. Obj. coefficient	
		=	40 – 24	
		=	₹ 16	(a)
	Allowable increase	=	Max. Obj. coefficient – Current value	
		=	60 – 40	
		=	₹ 20.00	(b)
Similarly, For x_2,	Allowable decrease	=	₹ 10.00	(c)
	Allowable increase	=	₹ 20.00	(d)

For example, if coefficient of x_1 is increased to 48, then the increase is 48 – 40 = ₹ 8.00 From (b), the allowable increase is 20, thus the increase in x_1 coefficient is 8/20 = 0.40 or 40%.

Similarly,

If coefficient of x_2 is decreased to 27, then the decrease is 30 – 27= ₹ 3.00.

From (c), the allowable decrease is 10, thus the decrease in x_2 coefficient is 3/10 = 0.30 or 30%.

Therefore, the percentage of increase in x_1 and the percentage of decrease in x_2 is 40 and 30 respectively.

i.e. 40% + 30% = 70%

For all the objective function coefficients that are changed, sum the percentage of the allowable increase and allowable decrease. If the sum of the percentages is less than or equal to 100%, the optimal solution does not change, i.e., x_1 and x_2 values will not change.

But Z_{max} will change, i.e.,

= 48(12.50) + 27(25)

= ₹ 1275.00

If the sum of the percentages of increase and decrease is greater than 100%, a different optimal solution exists. A revised problem must be solved in order to determine the new optimal values.

Notes

3.3.2 Change in the Right-hand Side Constraints Values and Effect on Optimal Solution

Suppose an additional 40 kgs of material 3 is available, the right-hand side constraint increases from 150 to 190 kgs.

Now, if the problem is solved, we get the optimal values as

$x_1 = 23.61$, $x_2 = 16.11$ and $Z_{max} = 1427.78$

From this, we can infer that an additional resources of 40 kgs increases the profit by = 1427.78 – 1250 = ₹ 177.78

Therefore, for one kg or one unit increase, the profit will increase by

= 177.78/40

= ₹ 4.44

Dual price is the improvement in the value of the optimal solution per unit increase in the right-hand side of a constraint. Hence, the dual price of material 3 is ₹ 4.44 per kg.

Increase in material 2 will simply increase the unused material 2 rather than increase in objective function. We cannot increase the RHS constraint values or the resources. If the limit increases, there will be a change in the optimal values.

The limit values are given in Table 2.10, i.e., Min RHS and Max RHS values.

For example, for material 3, the dual price ₹ 4.44 applies only to the limit range 150 kgs to 262.50 kgs.

Where there are simultaneous changes in more than one constraint RHS values, the

100 Per cent Rule must be applied.

Reduced Cost

$$\text{Reduced cost/unit of activity} = \begin{pmatrix}\text{Cost of consumed}\\ \text{resources per unit}\\ \text{of activity}\end{pmatrix} - \begin{pmatrix}\text{Profit per unit}\\ \text{of activity}\\ \\ \end{pmatrix}$$

If the activity's reduced cost per unit is positive, then its unit cost of consumed resources is higher than its unit profit, and the activity should be discarded. This means that the value of its associated variable in the optimum solution should be zero.

Alternatively, an activity that is economically attractive will have a zero reduced cost in the optimum solution signifying equilibrium between the output (unit profit) and the input (unit cost of consumed resources).

In the problem, both x_1 and x_2 assume positive values in the optimum solution and hence have zero reduced cost.

Considering one more variable x_3 with profit ₹ 50

$$Z_{max} = 40x_1 + 30x_2 + 50x_3$$

Subject to constraints,

$$4x_1 + 5x_2 + 6x_3 \le 175 \qquad \text{(a)}$$

$$2x_2 + 1x_3 \le 50 \qquad \text{(b)}$$

Notes

$$6x_1 + 3x_2 + 3x_3 \leq 150 \qquad \text{(c)}$$

where $x_1, x_2, x_3 \geq 0$

The sensitivity analysis of the problem is shown in the computer output below in Figure 3.2.

Figure 3.2: Sensitivity Analysis

The reduced cost indicates how much the objective function coefficient for a particular variable would have to improve before that decision function assumes a positive value in the optimal solution.

The reduced cost of ₹ 12.50 for decision variable x_2 tells us that the profit contribution would have to increase to at least 30 + 12.50 = 42.50 before x_3 could assume a positive value in the optimal solution.

3.4 Special Cases in Linear Programming

Let us discuss special cases in linear programming such as infeasibility and unboundedness. We will examine here how these special problems can be recognised while solving linear programming problems by the simplex method.

3.4.1 Multiple or Alternative Optimal Solutions

In certain conditions, a given LPP may have more than one solution yielding the same optimal function value. Each of such optimal solutions is termed as alternative optimal Solutions.

Example:

Maximise $\qquad$ 'Z' = $3x_1 + 2x_2$

Notes

Sub. to $x_1 \leq 40$

$x_2 \leq 60$

$3x_1 + 2x_2 \leq 180$

$x_1, x_2 \geq 0$

Solution:

Maximise $'Z' = 3x_1 + 2x_2$

Sub. to $x_1 + S_1 = 40$

$x_2 + S_2 = 60$

$3x_1 + 2x_2 + S_3 = 180$

First Iteration

BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min. Ratio
S_1	0	40	1	0	1	0	0	40/1 = 40 (KR) ?
S_2	0	60	0	1	0	1	0	–
S_3	0	180	3	2	0	0	1	180/3 = 60
		Z_j	0	0				
		C_j	3	2				
		$Z_j - C_j$	–3	–2				

Second Iteration

BV	C_B	X_B	X_1	X_2	Min. Ratio
X_1	3	40	1	0	–
S_2	0	60	0	1	60/1 = 60
S_3	0	180–40 (3) = 60	3–1 (3) = 0	2–0 (3) = 2	60/2 = 30 (KR) ?
		Z_j	3	0	
		C_j	3	2	
		$Z_j - C_j$	0	–2	

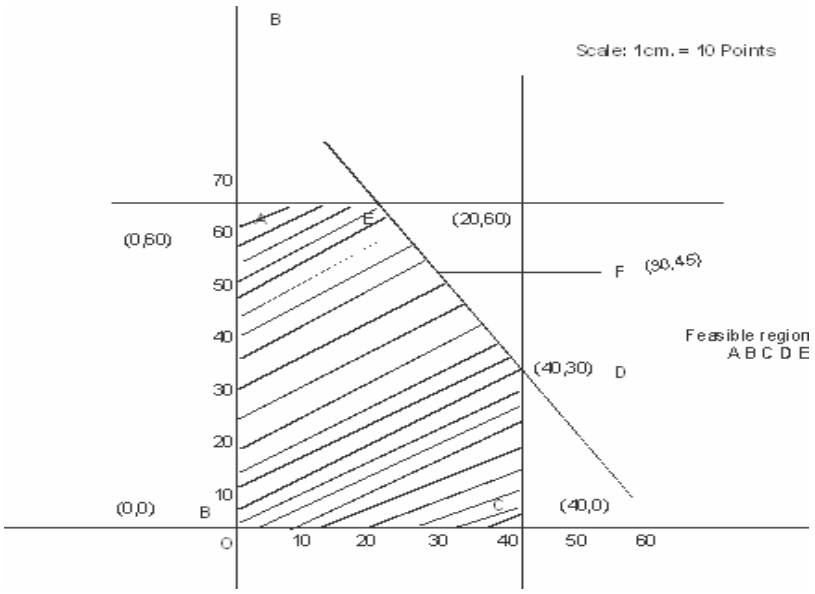
Third Iteration

BV	C_B	X_B	X_1	X_2	Min. Ratio
x_1	3	40	1	0	–
S_2	0	60–30 (1) = 30	0–0 (1) = 0	1–1 (0) = 0	–
S_3	0	60/2 = 30	0	2/2 = 1	–
		Z_j	3	2	
		C_j	3	2	
		$Z_j - C_j$	0	0	

Hence, optimum value for Z = 180 at $x_1 = 40$ and $x_2 = 30$. A graphical representation shows that the points D = (40, 30) and E = (20, 60) are optimal with Z = 180 units. Also, observe that every point on the line DE is optimal. For example, F (30, 45) gives Z = 180 units. Hence, the problem has infinite feasible solutions which are called alternative non-basic feasible solutions.

Notes

Figure 3.3: Graphical Representation of Alternative Non-basic Feasible Solutions



Example:

Maximise $'Z' = 3x_1 + 2x_2$

Sub. to, $x_1 \leq 4$

$x_2 \leq 6$

$3x_1 + 2x_2 \geq 18$

$x_1, x_2 \geq 0$

Solution:

Maximise $'Z' = 3x_1 + 2x_2$

Sub. to $x_1 + S_1 = 4$

$x_2 + S_2 = 6$

$3x_1 + 2x_2 + S_3 = 18$

Where, S_1, S_2 and S_3 are slack variables.

First Iteration

BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min. Ratio
S_1	0	4	$\boxed{1}$	0	1	0	0	4/1 = 4 (KR) ?
S_2	0	6	0	1	0	1	0	–
S_3	0	18	3	2	0	0	1	18/3 = 6
		Z_j	0	0				
		C_j	3	2				
		$Z_j - C_j$	–3	–2				

Notes

Second Iteration

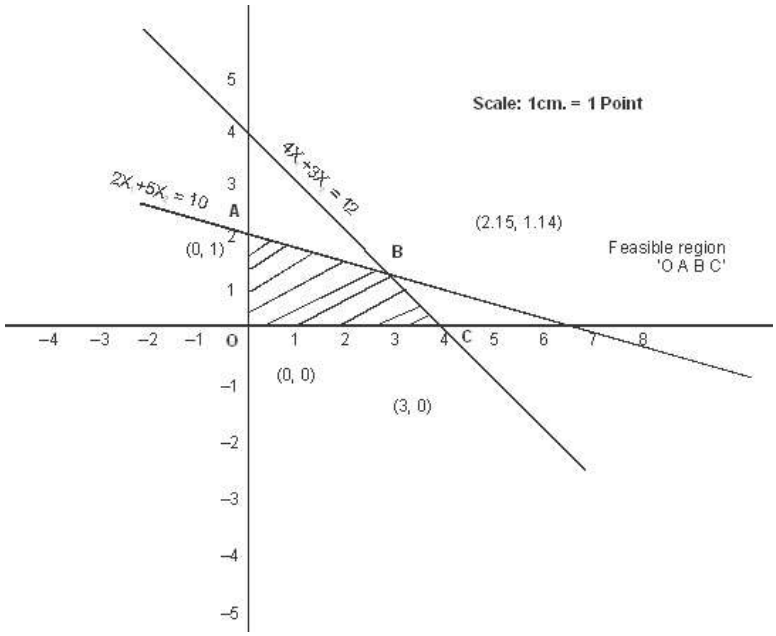
BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min. Ratio
x_1	0	4	1	0	–	–	–	–
S_2	0	6	0	1	–	–	–	6/1 = 6
S_3	0	18–4 (3) = 6	3–1 (3) = 0	2–0 (3) = 2	–	–	–	6/2 = 3 (KR) →
		Z_j C_j	3 3	0 2				
		$Z_j - C_j$	0	–2				

Third Iteration

BV	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min. Ratio
X_1	3	4	1	0	–	–	–	–
S_2	0	6–3 (1) = 3	0	1–1 (1) = 0	–	–	–	–
x_2	2	6/2 = 3	0	2/2 = 1	–	–	–	–
		Z_j C_j	3 3	2 2				
		$Z_j - C_j$	0	0				

The solution is optimal with Z = 18 at $x_1 = 4$ and $x_2 = 3$. A graphical representation of the problem reveals the multiple optimal solutions for the LPP.

Figure 3.4: Graphical Representation of Multiple Optimal Solutions for the LPP



3.4.2 Unbounded Solutions

Sometimes an LP problem will not have a finite solution. This means when no one or more decision variable values and the value of the objective function (maximization case) are permitted to increase infinitely without violating the feasibility condition, then the solution is said to be unbounded.

Notes

Example:

Maximise $\quad$ 'Z' = $3x + 2y$

Sub. to $\quad$ $x - y \leq 15$

$2x - y \leq 40$

$x, y \geq 0$

Where, S_1 and S_2 are slack variables.

First Iteration

BV	C_B	X_B	X	y	S_1	S_2	Min. Ratio
S_1	0	15	$\boxed{1}$	–1	1	0	15/1 = 15 (KR) ?
S_2	0	20	2	–1	0	1	20/2 = 10
		Z_j	0	0			
		C_j	3	2			
		$Z_j - C_j$	–3	–2			

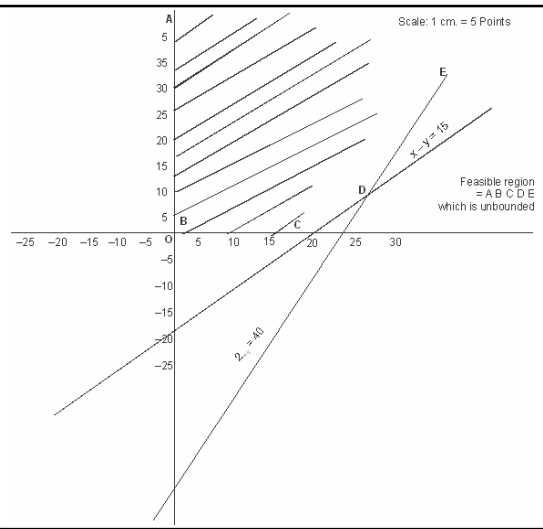
Second Iteration

BV	C_B	X_B	X	Y	S_1	S_2	Min. Ratio
S_1	0	15–10 (1) = 5	1–1 (0) = 0	–1+1/2 (1) –0.5	–	–	y' s are negative
X	3	10	1	–1/2	–	–	
		Z_j	3	–1.5			
		C_j	3	2			
		$Z_j - C_j$	0	–3.5			

At the end of second iteration y should enter the basis to improve the solution of Z. But there is no vector leaving the basis since y's are negative. Hence, the solution is unbounded optimum solution.

Note that the unbounded feasible region by graph is ABCDE. As the point A and E are extended, the value of Z increases. Hence, the solution is unbounded optimal solution.

Figure 3.5: Graphical Representation of Unbounded Optimal Solution



Notes

Example:

Maximise $'Z' = 4x_1 + x_2 + 3x_2 + 5x_4$

Sub. to $-4x_1 + 6x_2 + 5x_3 - 4x_3 \leq 20$

$3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$

$8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20$

$x_1, x_2, x_3, x_4 \geq 0$

Solution:

Maximise $'Z' = 4x_1 + x_2 + 3x_3 + 5x_4$

$-4x_1 + 6x_2 + 5x_3 - 4x_4 + S_1 = 20$

$3x_1 - 2x_2 + 4x_3 + x_4 + S_2 = 10$

$8x_1 - 3x_2 + 3x_3 + 2x_4 + S_3 = 20$

Where, S_1, S_2 and S_3 are slack variables.

First Iteration

BV	C_B	X_B	X_1	X_2	x_3	x_4	S_1	S_2	S_3	Min. Ratio
S_1	0	20	–4	6	5	–4	–	–	–	
S_2	0	10	3	–2	4	1	–	–	–	10/1 = 10
S_3	0	20	8	–3	3	2	–	–	–	20/2 = 10
		Z_j	0	0	0	0	0			
		C_j	4	4	1	3	5			
		$Z_j - C_j$	–0	4	–1	–3				

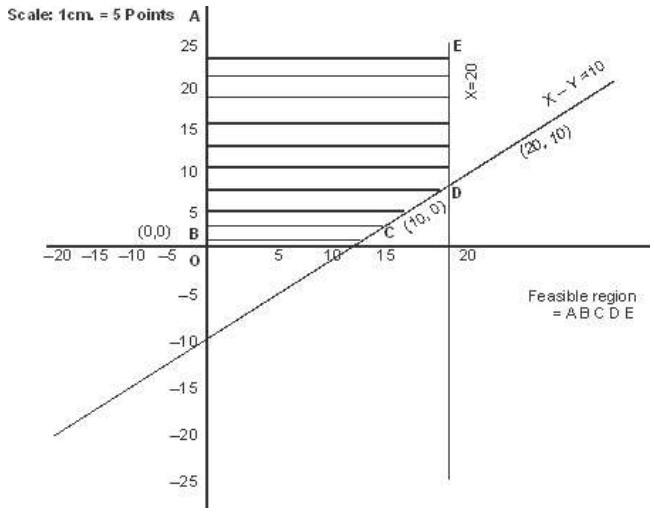
Second Iteration

BV	C_B	X_B	x_1	x_2	x_3	X_4	Min. Ratio
S_1	0	60	12	0	11	0	
S_2	0	10	–1	–0.5	2.5	0	10/1 = 10
S_3	0	10	4	–1.5	1.5	1	20/2 = 10
		Z_j	20	–7.5	7.5	5	
		C_j	4	1	3	5	
		$Z_j - C_j$	16	–8.5	4.5	0	

At the end of second iteration, observe that x_5 should enter the basis. But there is no chance for outgoing vector from the basis since y's are zero or negative. Hence, the solution is unbounded.

Notes

Figure 3.6: Graphical Representation of unbounded solution



3.4.3 Infeasibility

Infeasibility is a condition that arises when constraints are inconsistent (mutually exclusive) i.e. no value of the variables satisfy all the constraint simultaneously. This results in infeasible solution. If two or more constraints of a linear programming problem are mutually conflicting, it does not have a feasible solution. Let us take a problem to illustrate infeasibility.

Example: The Reddin Hardware Ltd. is producing two products, A and B. The profit contribution of product A is ₹ 5 per unit and of product B ₹ 4 per unit. Both the products go through the processing and assembly departments. Product A takes two minutes in the processing department and two minutes in the assembly department. Product B takes one minute in the processing department and one and a half minute in the assembly department. The maximum capacity of the processing department is 10,000 worker-minutes and of the assembly department 12,000 worker-hours. The marketing department has informed that a contract has been made with a hardware chain store for the supply of 6,500 units of product A and that there is no other demand for this product. There is no marketing constraint in the case of product B. What is the optimum product mix for the company?

The linear programming mode for the Reddin Hardware is as follows:

Let

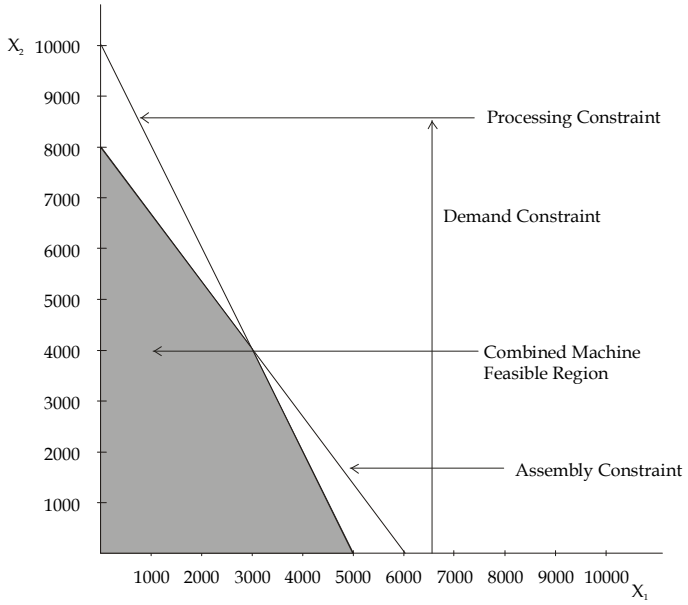
x_1 = product A

x_2 = product B

Maximise	$'Z' = 5x_1 + 4x_2$	Objective Function
	$2x_1 + 1x_2 \leq 10{,}000$	Processing Constraint
	$2x_1 + x_2 \leq 12{,}000$	Assembly Constraint
	$x_1 = 6{,}500$	Demand Constraint
	$x_1 \geq 0$	Non-negativity Constraint

Notes

Figure 3.7: The Reddin Problem—Infeasibility



The graphical presentation of the Reddin Problem in Figure shows that the combined machine feasibility region does not permit the production of 6,500 units of product A. Thus, the demand constraint and the machine capacity constraints are mutually conflicting, resulting in infeasibility.

Let us try to solve this problem by the simplex method. It is formulated below in the standard simplex form.

Row 1: $P- \quad 5x_1 - 4x_2 \quad -50a_1 = 0$

Row 2: $2x_1 + 1x_2 + s_1 = 10{,}000$

Row 3: $2x_1 + \frac{3}{2}x_2 + s_2 = 12{,}000$

Row 4: $x_1 + a_1 = 6{,}500$

Artificial variable a1 has been assigned a coefficient of 50 in the objective function.

Towards The First Basis

Row 4 × 50: $50x_1 + 50a_1 = 325{,}000$

Old Row 1: $Z - 5x_1 - 4x_2 - 50a_1 = 0$

New Row 1: $Z + 45x_1 - 4x_2 = 325{,}000$

Table 3.2: The First Basis

Row 1: P	$+45x_1$	$-4x_2$				$= 325{,}000$
Row 2:	$2x_1$	$+x_2$	$+s_1$			$= 10{,}000$
Row 3:	$2x_1$	$+x_2$		$+s_2$		$= 12{,}000$
Row 4:	x_1				$+a_1$	$= 6{,}500$

Notes

Table 3.3: The Second Basis

Row 1: P +	x_1			$- s_2$		= 357,000
Row 2:	x_1		$+ s_1$	$- s_2$		= 2,000
Row 3:	x_1	$+ x_2$		$+s_2$		= 8,000
Row 4:	x_1				$+a_1$	= 6,500

Since there is no variable with a negative coefficient in Row 1, no further improvement in profit is possible. But artificial variable a1 continues to exist in Row 4 and it is also basic variable. This means that the optimum solution has not been reached. The problem has no solution because of infeasibility.

Case Study

Linear Programming & Technical Accounting

A company manufactures 2 products a 10 cu. Ft & 6 cu. ft. refrigerator. The demand for the former has been estimated at 15,000 and for the later at 24,000.

The production process is broken down into 3 stages:

1. Shell production
2. Motor assembly
3. Refrigerator assembly

The production manager estimates that the shell production unit can through put up to 36,000 smaller refrigerators, but only half of this quantity of the 10 cu. ft. product. Similarly, the motor assembly unit can through put 30,000 smaller motors or 80% of this quantity of the other product. The assembly department has no restrictions as labour unskilled is required. The accounting function has produced the standard cost breakdown as follows:

Standard Costs of Production

(in Dollars)

Items	10 cu. ft	6 cu. ft.
Direct materials	30	24
Direct labour	16	10
Variable overhead	14	11
Marginal cost	60	45
Selling price	90	65
Contribution	30	20

Fixed overhead charges are estimated as $ 4,00,000.

Traditional accounting would entail the manufacture of 15,000 of the 10 cu ft. refrigerators before the shell production unit capacity is exceeded only 6,000 smaller refrigerator would be manufactured. On the basis of direct labour, $3,20,000 of fixed overheads would be attributed to 10 cu. ft. & 80,000 to the 6 cu. ft.

Contd...

Notes

The net profit would thus be

Items	10 cu. ft	6 cu. Ft.	Total (in '000 dollars)
Contribution	450	120	570
Overhead allocation	320	80	400
Net profit	130	40	170

But this is an over simplification when a resource is limiting. The scarce resource is shell production and contribution per unit is $ 20 for the cu. ft & $ 15 for the 10 cu. ft. This suggests that the 6 cu. ft. is more profitable and should be produced to its demand of 24,000. For this, the net profit would stand at $2,24,000.

But both are misleading because profitability is not directly related to contribution per unit product or per unit resource. Hence an examination from the linear programming view point was carried out to arrive at a logical solution.

Questions:

1. Formulate the given problem as a linear programming problem.
2. Solving the problem by the revised simplex technique to determine the maximum profit.

Self Assessment

Fill in the blanks:

8. The optimal solution of the primal problem appears under the ..... variables in the ..... row of the final simplex table associated with the dual problem.
9. The ..... analysis involves "what if" questions.
10. The original linear programming problem is known as ..... problem.

3.5 Summary

In this unit, you learned the mechanics of obtaining an optimal solution to a linear programming problem by the simplex method. The simplex method is an appropriate method for solving a ≤ type linear programming problem with more than two decision variables. Two phase and M-method are used to solve problems of ≥ or ≤ type constraints. Further, the simplex method can also identify multiple, unbounded and infeasible problems.

3.6 Keywords

Artificial Variables: Temporary slack variables which re used for calculations.

Simplex Method: A method which examines the extreme points in a systematic manner, repeating the same set of steps of the algorithms until an optimal solution is reached.

Slack Variables: Amount of unused resources.

Surplus Variables: A surplus variable represents the amount by which solution exceeds a resource.

Unconstrained Variable: The variable having no non-negativity constraint.

Notes

3.7 Review Questions

1. Explain the simplex procedure to solve the Linear Programming Problem.
2. How do you recognize optimality in the simplex method?
3. Can a vector that is inserted at one iteration in simplex method be removed immediately at the next iteration? When can this occur and when is it impossible?
4. Explain briefly the term 'Artificial' variables.
5. Explain the use of artificial variables in L.P..
6. Maximize: $Z = x_1 + x_2$

   Subject to $x_1 + 5x_2 \leq 5$

   $2x_1 + x_2 \leq 4$ with x_1, x_2 non-negative.
7. Minimize: $Z = 3x_1 + 4x_2$

   Subject to $2x_1 + x_2 \leq 6$

   $2x_1 + 3x_2 \geq 9$

   With, x_1, x_2 non-negative.
8. Minimize: $Z = x_1 + 2x_2$

   Subject to $x_1 + 3x_2 \leq 11$

   $2x_1 + x_2 \geq 9$

   With, x_1, x_2 non-negative.
9. Maximize $Z = -x_1 - x_2$

   Subject to $x_1 + 2x_2 \geq 5000$

   $5x_1 + 3x_2 \geq 12000$

   With, x_1, x_2 non-negative
10. Maximize $Z = 2x_1 + 3x_2 + 4x_3$

    $x_1 + x_2 + x_3 \geq 1$

    $x_1 + x_2 + 2x_3 = 2$

    $2x_1 + 2x_2 + x_3 \geq 4$

    With all variables non-negative
11. Minimize: $Z = 14x_1 + 13x_2 + 11x_3 + 13x_4 + 13x_5 + 12x_6$

    Subject to $x_1 + x_2 + x_3 = 1200$

    $x_4 + x_5 + x_6 = 1000$

    $x_1 + x_4 = 1000$

    $x_2 + x_5 = 700$

    $x_3 + x_6 = 500$

    With all variables non-negative.

Notes

12. Discuss the role of sensitivity analysis in linear programming.
13. In sensitivity analysis, explain
    (a) The effect of change of objective function coefficients.
    (b) The effect of change in the right hand side of constraints.

Answers: Self Assessment

1.	z = 21	2.	z = 28
3.	key element or pivotal element	4.	slack
5.	surplus	6.	z = 2,100
7.	z = 7.45 – 3.51m	8.	slack, last
9.	sensitivity	10.	Primal

3.8 Further Readings

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Unit 4: Linear Programming – Duality

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4.3 Advantages of Duality

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Objectives

After studying this unit, you will be able to:

- Analyze the importance of duality in linear programming
- Understand dual formation and solution
- Learn the significance and advantages of duality in LPP

Introduction

For every linear programming problem, there is an associated linear programming problem. The former problem is called primal and the latter is called its dual and vice versa. The two problems may appear to have superficial relationship between each other but they possess very intimately related properties and useful one, so that the optimal solution of one problem gives complete information about the optimal solution to the other. In other words, the optimal solutions for both the problems are same.

The concept of duality is very much useful to obtain additional information about the variations in the optimal solutions when certain changes are effected in the constraint coefficients, resource availabilities and objective function coefficients. This is termed as post-optimality or sensitivity analysis.

4.1 Concept of Duality

One part of a Linear Programming Problem (LPP) is called the *Primal* and the other part is called the *Dual*. In other words, each maximization problem in LP has its corresponding problem, called the dual, which is a minimization problem. Similarly, each minimization problem has its corresponding dual, a maximization problem. For example, if the primal is concerned with maximizing the contribution from the three products A, B, and C and from the three departments X, Y, and Z, then the dual will be concerned with minimizing the costs associated with the time used in the three departments to produce those three products. An optimal solution from the primal and the dual problem would be same as they both originate from the same set of data.

Notes

Caution The necessary and sufficient condition for any LPP and its dual to have optimal solutions is that both have feasible solutions.

4.2 Dual Formulation

The following procedure is adopted to convert primal problem into its dual. Simplex method is applied to obtain the optimal solution for both primal and dual form.

Step 1: For each constraint, in primal problem there is an associated variable in the dual problem.

Step 2: The elements of right-hand side of the constraints will be taken as the coefficients of the objective function in the dual problem.

Step 3: If the primal problem is maximization, then its dual problem will be minimization and vice versa.

Step 4: The inequalities of the constraints should be interchanged from ³ to ≤ and vice versa and the variables in both the problems are non-negative.

Step 5: The rows of primal problem are changed to columns in the dual problem. In other words, the matrix A of the primal problem will be changed to its transpose (A) for the dual problem.

Step 6: The coefficients of the objective function will be taken to the right hand side of the constraints of the dual problem.

Example 1: Write the dual of the following problem.

Maximise $\quad 'Z' = -6x_1 + 7x_2$

Subject to, $\quad -x_1 + 2x_2 \le -5$

$3x_1 + 4x_2 \le 7$

$x_1, x_2 \ge 0$

Solution: The given problem is considered as primal linear programming problem. To convert it into dual, the following procedure is adopted.

Step 1: There are 2 constraints and hence the dual problem will have 2 variables. Let us denote them as y_1 and y_2.

Step 2: The right hand side of the constraints are –5 and 7 which are taken as the coefficients of the variables y_1 and y_2 in the objective function.

Step 3: The primal objective problem is maximization and hence the dual seeks minimization for the objective function. Hence, the objective functions for the dual problem is given by

$$\text{Minimise } Z = -5y_1 + 7y_2$$

Step 4: The inequalities of the constraints for the primal problem are of the type (≤). Hence, the inequalities for the dual constraints will be of the type (≥).

Step 5: The coefficient matrix for the primal problem is

$$A = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$$

Notes

The transpose of this matrix which serves as the coefficient matrix for the dual problem is given by.

$$A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$$

Step 6: The coefficients of the objective function for the given primal are –6 and 7. They are taken on the right hand side of the constraints for the dual problem. Hence, the constraints for the dual problem are represented as

$$-y_1 + 3y_2 \geq -6$$
$$2y_1 + 4y_2 \geq 7$$
$$y_1, y_2 \geq 0$$

Example 2: Find the dual of the following problem:

Minimise $Z = 3x_1 + 4x_2 + 5x_3$

Subject to $x_1 + x_3 \geq 3$

$x_2 + x_3 \geq 4$

$x_1, x_2, x_3 \geq 0$

Solution: Let the given problem be denoted as primal. The objective of the primal is minimization and hence the objective of the dual is maximization. There are two constraints for the primal problem. Hence, the problem has 2 variables namely, y_1 and y_2. The right hand side of the constraints are 3 and 4 which are the coefficients for y_1 and y_2 in the objective function for the dual problem given as

Maximise $Z = 3y_1 + 4y_2$

The coefficient of the objective function in the primal problem are 3, 4 and 5 which serve as the right hand side of the constraints for the dual problem. The inequalities of the constraints of the type (≥) are converted as (≤) type for the dual problem. The coefficient matrix for primal problem is

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

And hence the coefficient matrix for the dual problem is

$$A^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Hence, the dual problem is represented as

Maximise 'Z' = $3y_1 + 4y_2$

Subject to

$y_1 + y_2 \leq 3$	is	$y_1 \leq 3$
$y_1 + y_2 \leq 4$		$y_2 \leq 4$
$y_1 + y_2 \leq 5$		$y_1 + y_2 \leq 5$
$y_1, y_2 \geq 0$		$y_1, y_2 \geq 0$

Notes

Example 3: Write the dual of the following problem. Show that the optimal solution for primal and dual are same.

Minimise $'Z' = 4x_1 + 3x_2 + 6x_3$

Subject to $x_1 + x_3 \geq 2$

$x_2 + x_3 \geq 5$

$x_1, x_2, x_3 \geq 0$

Solution:

The dual for the given problem can be written as

Maximise $Z = 2y_1 + 5y_2$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Subject to $y_1 \leq 4$

$y_2 \leq 3$

$y_1 + y_2 \leq 6$

$$A^{|} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$y_1, y_2 \geq 0$

Let us use simple technique to obtain the optimum solution.

Maximise $'Z' = 2y_1 + 5y_2$

Subject to $y_1 + y_3 = 4$

$y_2 + y_4 = 3$

$y_1 + y_2 + y_5 = 6$

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Where, y_3, y_4 and y_5 are slack variables.

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	4	1	0	1	0	0	–
S_2	0	3	0	$\boxed{1}$	0	1	0	3/1 = 3 (KR) ?
S_3	0	6	1	1	0	0	1	6/1 = 6
		Z_J C_J	0 2	0 5	0 0	0 0	0 0	
		$Z_J - C_J$	–2	–5	0	0	0	

(↑ KC)

Notes

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	4	1	0	1	0	0	4/1 = 4
y_2	5	3	0	1	0	1	0	––
S_3	0	6–3 (1)= 3	1–0 (1)= $\boxed{1}$	1–1 (1)= 0	0–0 (1)= 0	0–1 (1)= –1	1–0 (1)= 1	3/1 = 3 KR→
		Z_J C_J	0 2	5 5	0 0	5 0	0 0	
		$Z_J - C_J$	–2	0	0	5	0	

(↑ KC)

BV	C_B	X_B	Y_1	Y_2	S_1	S_2	S_3	Min. Ratio
S_1	0	4–3 (1) = 1	1–1 (1) = 0	0–0 (1) = 0	1–0 (1) = 1	0+1 (1) = 1	0–1 (1) = –1	-
y_2	5	3	0	1	0	1	0	-
y_1	2	3	1	0	0	–1	1	-
		Z_J C_J	2 2	5 5	0 0	3 0	2 0	
		$Z_J - C_J$	0	0	0	3	2	

Z = 15 + 6

Z = 21

$Z_J - C_J = 0$ for all V_j, hence the solution is optimal with Z = 21 at $y_1 = 3$ and y_2 for the dual form. Now, let us take the primal form and obtain the optimal solution.

Minimise $Z = 4x_1 + 3x_2 + 6x_2$

Subject to $x_1 + x_3 \geq 2$

$x_2 + x_3 \geq 5$

$x_1, x_2, x_3 \geq 0$

Maximise $Z' = -4x - 3x_2 - 6x_3$

Subject to $x_1 + x_3 - x_4 + a_1 = 2$

$x_2 + x_3 - x_5 + a_2 = 5$

Where x_4 and x_5 are surplus variables; a_1 and a_2 are artificial variables.

BV	C_B	X_B	y_1	y_2	y_3	S_1	S_2	a_1	a_2	Min. Ratio
a_1	–M	2	1	0	$\boxed{1}$	–1	0	1	0	2/1 = 2 KR→
a_2	–M	5	0	1	1	0	–1	0	1	5/1 = 5
		Z_j C_j	–M –4	–M –3	–2M –6					
		$Z_j - C_j$	–M+4	–M+3	–2M+6					

(↑ KC)

Z = –7M

Notes

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	a_1	a_2	Min Ratio
y_3	–6	2	1	0	1	-	-	-	-	-
a_2	M	5–2 (1) = 3	0–1 (1)= –1	1–0 (1)= $\boxed{1}$	1–1 (1)= 0	-	-	-	-	4/1 = 4→
		Z_j C_j	M–6 –4	–M –3	–6 –6					
		$Z_j - C_j$	M–2	–M+3	0					

(↑ KC)

Z = –12 – 3m

BV	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	a_1	a_2	Min Ratio
y_3	–6	2	1	0	1	–	–	–	–	–
y_2	–3	3	–1	1	0	–	–	–	–	–
		Z_j C_j	–3 –4	–3 –3	–6 –6					
		$Z_j - C_j$	1	0	0					

$Z_J - C_J \geq 0$ hence, solution is optimal.

Maximise 'Z_1' = –21

Therefore Minimise Z = 21

Hence, the optimal solution for primal problem is Z = 21, where $x_1 = 0$, $x_2 = 3$ and $x_3 = 2$. Observe that the optimal solution for primal and dual is same i.e., Z = 21

Primal Form
Min. $Z = 4x_1 + 3x_2 + 6x_3$
Min. Z = 21
$x_1 = 0$
$x_2 = 3$
$x_3 = 2$

Dual Form
Max. $Z = 2y_1 + 5y_2$
Max. Z = 21
$y_1 = 3$
$y_2 = 3$

The primal solution can be obtained from the last table of the dual problem without simplex iteration. Notice that $Z_j - C_j$ entries for S_1, S_2 and S_3 are respectively 0, 3 and 2 in the final iteration of the dual problem there are optimum value of the variables in the primal problem. Hence, $x_1 = 0$, $x_2 = 3$ and $x_3 = 2$ and Z = 21 which tallies with the solution obtained by simplex iteration of the primal problem.

Notes The study of duality is very important in LP. Knowledge of duality allows one to develop increased insight into LP solution interpretation. Also, when solving the dual of any problem, one simultaneously solves the primal. Thus, duality is an alternative way of solving LP problems. However, given today's computer capabilities, this is an infrequently used aspect of duality. Therefore, we concentrate on the study of duality as a means of gaining insight into the LP solution.

Notes

Task Find the dual to following problem:

Maximise $Z = 4x_1 + 5x_2 + 3x_3 + 6x_4$

Subject to $x_1 + 3x_2 + x_3 + 2x_4 \leq 2$

$3x_1 + 3x_2 + 2x_3 + 2x_4 \leq 4$

$3x_1 + 2x_2 + 4x_3\ 5x_4 \leq 6$

$x_1, x_2, x_3, x_4 \geq 0$

4.3 Advantages of Duality

1. It is advantageous to solve the dual of a primal having less number of constraints because the number of constraints usually equals the number of iterations required to solve the problem.
2. It avoids the necessity for adding surplus or artificial variables and solves the problem quickly (the technique is known as the primal-dual method).
3. The dual variables provide an important economic interpretation of the final solution of an LP problem.
4. It is quite useful when investigating changes in the parameters of an LPP (the technique known as sensitivity analysis).
5. Duality is used to solve an LPP by the simplex method in which the initial solution is infeasible.

Did u know? The dual of the dual problem is again the primal problem.

Task Find out the managerial significance of Duality with the help of some real life cases.

Self Assessment

State true or false:

1. One part of a Linear Programming Problem (LPP) is called the *Primal* and the other part is called the *Dual*.
2. In the primal problem, the objective function is a exponential combination of *n* variables.
3. In the dual problem, the dual vector multiplies the constants that determine the positions of the constraints in the primal.
4. Duality is quite useful when investigating changes in the parameters of an LPP(the technique known as sensitivity analysis).
5. It is advantageous to solve the dual of a primal having less number of constraints because the number of constraints usually equals the number of iterations required to solve the problem.

Notes

Case Study

Minimum Dietary Requirement

A dietician wishes to design a minimum-cost diet to meet minimum daily requirements for calories, protein, carbohydrate, fat, vitamin A and vitamin B dietary needs. Several different foods can be used in the diet, with data as specified in the following table.

	Content and costs per pound consumed						
	Food 1	*Food* 2	....	*Food* j	....	*Food* n	*Daily requirements*
Calories	a_{11}	a_{12}		a_{1j}		a_{1n}	b_1
Protein (grams)	a_{21}	a_{22}		a_{2j}		a_{2n}	b_2
Carbohydrate (grams)	a_{31}	a_{32}		a_{3j}		a_{3n}	b_3
Fat (grams)	a_{41}	a_{42}		a_{4j}		a_{4n}	b_4
Vitamin A (milligrams)	a_{51}	a_{52}		a_{5j}		a_{5n}	b_5
Vitamin B (milligrams)	a_{61}	a_{62}		a_{6j}		a_{6n}	b_6
Costs (dollars)	c_1	c_2		c_j		c_n	

Questions:

1. Formulate a linear program to determine which foods to include in the minimum cost diet. (More than the minimum daily requirements of any dietary need can be consumed.)
2. State the dual to the diet problem, specifying the units of measurement for each of the dual variables. Interpret the dual problem in terms of a druggist who sets prices on the dietary needs in a manner to sell a dietary pill with b_1, b_2, b_3, b_4, b_5 and b_6 units of the given dietary needs at maximum profit.

4.4 Summary

One part of a Linear Programming Problem (LPP) is called the *Primal* and the other part is called the *Dual*. In other words, each maximization problem in LP has its corresponding problem, called the dual, which is a minimization problem. Similarly, each minimization problem has its corresponding dual, a maximization problem. For example, if the primal is concerned with maximizing the contribution from the three products A, B, and C and from the three departments X, Y, and Z, then the dual will be concerned with minimizing the costs associated with the time used in the three departments to produce those three products. An optimal solution from the primal and the dual problem would be same as they both originate from the same set of data.

Notes

4.5 Keywords

Dual Problem: In the dual problem, the dual vector multiplies the constants that determine the positions of the constraints in the primal.

Duality principle: In optimization theory, the duality principle states that optimization problems may be viewed from either of two perspectives, the primal problem or the dual problem.

Primal Problems: In the primal problem, the objective function is a linear combination of *n* variables.

4.6 Review Questions

Determine the duals of the given problems.

1. Minimise $Z = 12x_1 + 26x_2 + 80x_3$

$$2x_1 + 6x_2 + 5x_3 \geq 4$$

$$4x_1 + 2x_2 + x_3 \geq 10$$

$$x_1 + x_2 + 2x_3 \geq 6$$

With all variables non-negative.

2. Minimise $Z = 3x_1 + 2x_2 + x_3 + 2x_4 + 3x_5$

Subject to: $2x_1 + 5x_2 + x_4 + x_5 \geq 6$

$$4x_2 - 2x_3 + 2x_4 + 3x_5 \geq 5$$

$$x_1 - 6x_2 + 3x_2 + 7x_4 + 5x_5 \geq 7$$

With all variables non-negative.

3. Maximise $Z = 6x_1 - x_2 + 3x_3$

Subject to $7x_1 + 11x_2 + 3x_3 \leq 25$

$$2x_1 + 8x_2 + 6x_3 \leq 30$$

$$6x_1 + x_2 + 7x_3 \leq 35$$

With all variables non-negative.

4. Minimise $Z = 10x_1 + 15x_2 + 20x_3 + 25x_4$

Subject to: $8x_1 + 6x_2 - x_3 + x_4 \geq 16$

$$3x_1 + 2x_3 - x_4 \geq 20$$

With all variables non-negative.

5. Minimise $Z = x_1 + 2x_2 + x_3$

Subject to $x_2 + x_3 = 1$

$$3x_1 + x_2 + 3x_3 = 4$$

With all variables non-negative.

Notes

Answers: Self Assessment

1. True
2. False
3. True
4. True
5. True

4.7 Further Readings

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Unit 5: Transportation Problem

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Objectives

After studying this unit, you will be able to:

- Understand the meaning of operations research
- Know about the history of operations research
- Discuss the scope and application of operations research
- Explain the various types of models used in operations research

Notes

Introduction

Transportation problem is a particular class of linear programming, which is associated with day-to-day activities in our real life and mainly deals with logistics. It helps in solving problems on distribution and transportation of resources from one place to another. The goods are transported from a set of sources (e.g., factory) to a set of destinations (e.g., warehouse) to meet the specific requirements. In other words, transportation problems deal with the transportation of a product manufactured at different plants (supply origins) to a number of different warehouses (demand destinations). The objective is to satisfy the demand at destinations from the supply constraints at the minimum transportation cost possible. To achieve this objective, we must know the quantity of available supplies and the quantities demanded. In addition, we must also know the location, to find the cost of transporting one unit of commodity from the place of origin to the destination. The model is useful for making strategic decisions involved in selecting optimum transportation routes so as to allocate the production of various plants to several warehouses or distribution centers.

The transportation model can also be used in making location decisions. The model helps in locating a new facility, a manufacturing plant or an office when two or more number of locations is under consideration. The total transportation cost, distribution cost or shipping cost and production costs are to be minimized by applying the model.

5.1 Modeling of Transportation Problem

A transportation problem can be expressed in two ways.

1. Mathematical representation
2. Network representation

Obviously the method used for solving the problems is the formulation of transportation problem through mathematical methods. But for understanding of the readers, network representation is equally important.

Let us understand each of them one by one.

5.1.1 Mathematical Representation

The transportation problem applies to situations where a single commodity is to be transported from various sources of supply **(origins)** to various demands **(destinations).**

Let there be *m* sources of supply S_1, S_2, …….....S_m having a_i (i = 1, 2,.....m) units of supplies respectively to be transported among *n* destinations D_1, D_2 ……D$_n$ with b_j (j = 1,2…..n) units of requirements respectively. Let C_{ij} be the cost for shipping one unit of the commodity from source i, to destination j for each route. If x_{ij} represents the units shipped per route from source i, to destination j, then the problem is to determine the transportation schedule which minimizes the total transportation cost of satisfying supply and demand conditions.

The transportation problem can be stated mathematically as a linear programming problem as below:

Minimize $$Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Notes

Subject to constraints,

$$\sum_{j=1}^{n} x_{ij} = a_{i,} \quad i = 1,2,\ldots..m \text{ (supply constraints)}$$

$$\sum_{i=1}^{n} x_{ij} = b_{j,} \quad j = 1,2,\ldots..m \text{ (demand constraints)}$$

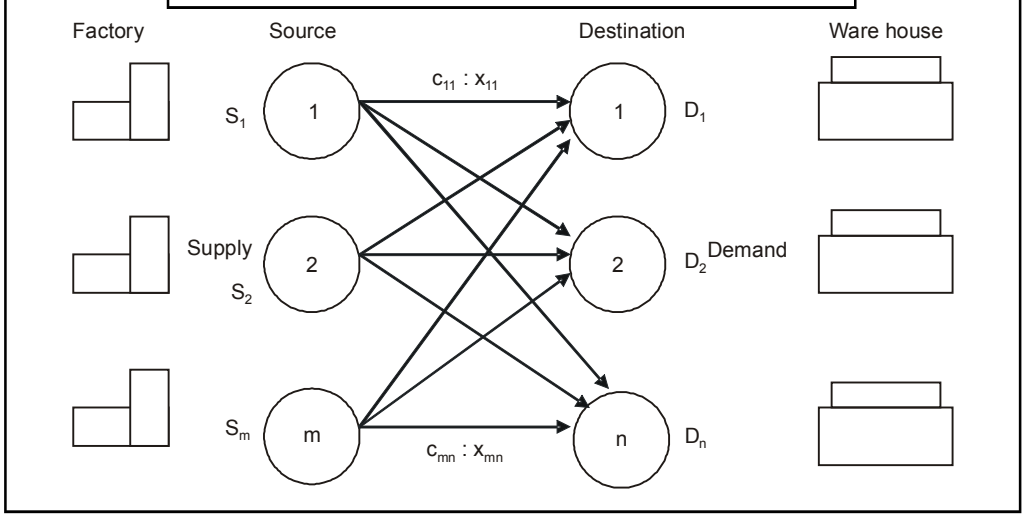
and x_{ij} ³ 0 for all *i = 1,2,…..m and,*

j = 1,2,…..m

5.1.2 Network Representation of Transportation Model

The transportation model is represented by a network diagram in Figure 3.1

Figure 5.1: Network Representation of Transportation Model



where,

m be the number of sources,

n be the number of destinations,

S_m be the supply at source m,

D_n be the demand at destination n,

c_{mn} be the cost of transportation from source m to destination n, and

x_{mn} be the number of units to be shipped from source m to destination n.

The objective is to minimize the total transportation cost by determining the unknowns x_{mn}, i.e., the number of units to be shipped from the sources and the destinations while satisfying all the supply and demand requirements.

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5.1.3 General Representation of Transportation Model

The Transportation problem can also be represented in a tabular form as shown in Table 5.1

Let c_{ij} be the cost of transporting a unit of the product from i^{th} origin to j^{th} destination.

a_i be the quantity of the commodity available at source i,

b_j be the quantity of the commodity needed at destination j, and

x_{ij} be the quantity transported from i^{th} source to j^{th} destination

Table 5.1: Tabular Representation of Transportation Model

From \ To	D_1	D_2	…	D_3	Supply a_1
S_1	C_{11} X_{11}	C_{12} X_{12}	…	C_{1n}	a_1
S_2	C_{21} X_{21}	C_{22} X_{22}	…	C_{2n}	a_2
. . .	. . .	. . .	… … …	. . .	. . .
S_m	C_{m1} x_{m1}	C_{m2} x_{m2}	…	C_{mn}	a_m
b_j Demand	b_1	b_2	…	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

If the total supply is equal to total demand, then the given transportation problem is a balanced one.

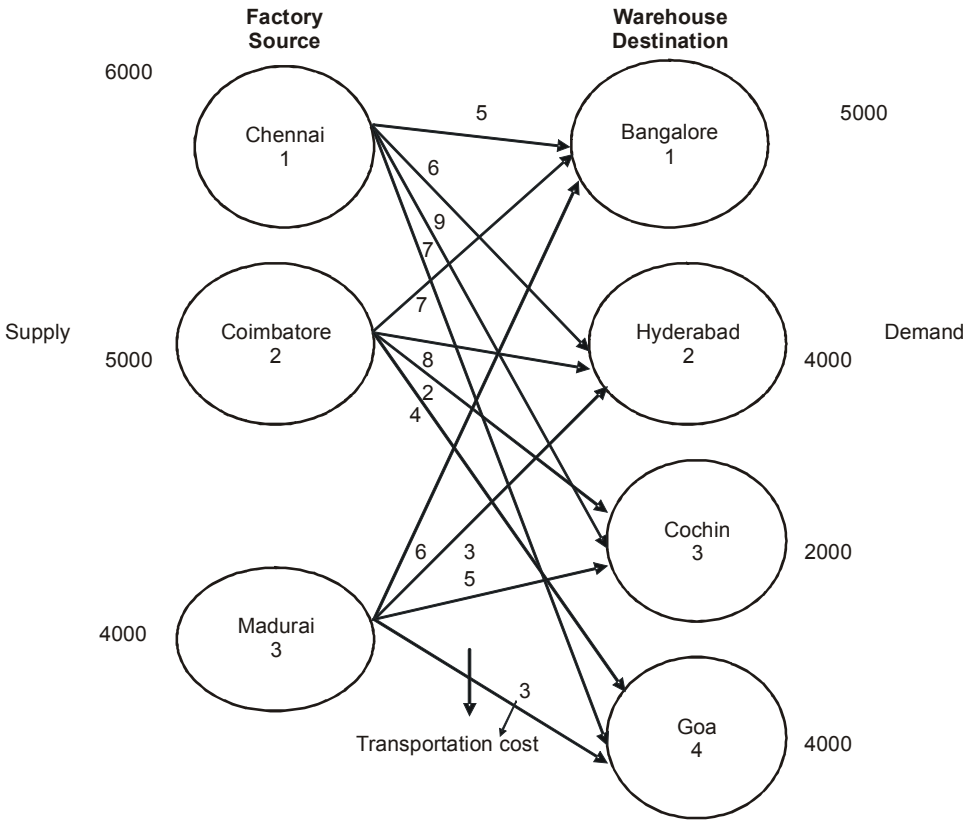
5.2 Use of Linear Programming to Solve Transportation Problem

The network diagram shown in Figure 5.2 represents the transportation model of GM Textiles units located at Chennai, Coimbatore and Madurai. GM Textiles produces ready-made garments at these locations with capacities 6000, 5000 and 4000 units per week at Chennai, Coimbatore and Madurai respectively. The textile unit distributes its ready-made garments through four of its wholesale distributors situated at four locations Bangalore, Hyderabad, Cochin and Goa. The weekly demand of the distributors are 5000, 4000, 2000 and 4000 units for Bangalore, Hyderabad, Cochin and Goa respectively.

The cost of transportation per unit varies between different supply points and destination points. The transportation costs are given in the network diagram.

Notes

Figure 5.2: Network Model of Transportation Problem



The management of GM Textiles would like to determine the number of units to be shipped from each textile unit to satisfy the demand of each wholesale distributor. The supply, demand and transportation cost are as follows:

Table 5.2 (a): Production Capacities

Supply	Textile Unit	Weekly Production (Units)
1	Chennai	6000
2	Coimbatore	5000
3	Madurai	4000

Table 5.2 (b): Demand Requirements

Destination	Wholesale Distributor	Weekly Demand (Units)
1	Bangalore	5000
2	Hyderabad	4000
3	Cochin	2000
4	Goa	4000

Notes

Table 5.2 (c): Transportation Cost per Unit

Supply	Destination			
	Bangalore	Hyderabad	Cochin	Goa
Chennai	5	6	9	7
Coimbatore	7	8	2	4
Madurai	6	3	5	3

A linear programming model can be used to solve the transportation problem.

Let,

X_{11} be number of units shipped from source1 (Chennai) to destination 1 (Bangalore).

X_{12} be number of units shipped from source1 (Chennai) to destination 2 (Hyderabad).

X_{13} number of units shipped from source 1 (Chennai) to destination 3 (Cochin).

X_{14} number of units shipped from source 1 (Chennai) to destination 4 (Goa).

:

and so on.

X_{ij} = number of units shipped from source i to destination j, where i = 1,2,……..m and,

j = 1,2,……n.

5.3 Minimizing Case

Objective Function

The objective is to minimize the total transportation cost. Using the cost data table, the following equation can be arrived at:

- Transportation cost for units shipped from Chennai = $5x_{11}+6x_{12}+9x_{13}+7x_{14}$
- Transportation cost for units shipped from Coimbatore = $7x_{21}+8x_{22}+2x_{23}+4x_{24}$
- Transportation cost for units shipped from Madurai = $6x_{31}+3x_{32}+5x_{33}+3x_{34}$

Combining the transportation cost for all the units shipped from each supply point with the objective to minimize the transportation cost, the objective function will be,

Minimize $Z = 5x_{11}+6x_{12}+9x_{13}+7x_{14}+7x_{21}+8x_{22}+2x_{23}+4x_{24}+6x_{31}+3x_{32}+5x_{33}+3x_{34}$

Constraints: In transportation problems, there are supply constraints for each source, and demand constraints for each destination.

Supply constraints:

For Chennai, $x_{11}+ x_{12}+ x_{13}+ x_{14} \leq 6000$

For Coimbatore, $x_{21}+ x_{22}+ x_{23}+ x_{24} \leq 5000$

For Madurai, $x_{31}+ x_{32}+ x_{33}+ x_{34} \leq 4000$

Demand constraints:

For B'lore, $x_{11}+ x_{21}+ x_{31} = 5000$

For Hyderabad, $x_{12} + x_{22} + x_{32} = 4000$

Notes

For Cochin, $x_{13} + x_{23} + x_{33} = 2000$

For Goa, $x_{14} + x_{24} + x_{34} = 4000$

The linear programming model for GM Textiles will be write in the next line. Minimize $Z = 5x_{11} + 6x_{12} + 9x_{13} + 7x_{14} + 7x_{21} + 8x_{22} + 2x_{23} + 4x_{24} + 6x_{31} + 3x_{32} + 5x_{33} + 3x_{34}$

Subject to constraints,

$X_{11}+x_{12}+x_{13}+x_{14} \leq 6000$ (i)

$X_{21}+x_{22}+x_{23}+x_{24} \leq 5000$ (ii)

$X_{31}+x_{32}+x_{33}+x_{34} \leq 4000$ (iii)

$X_{11}+x_{21}+x_{31} = 5000$ (iv)

$X_{12}+x_{22}+x_{32} = 4000$ (v)

$X_{13}+x_{23}+x_{33} = 2000$ (vi)

$X_{14}+x_{24}+x_{34} = 4000$ (vii)

Where, $x_{ij} \geq 0$ for i = 1, 2, 3 and j = 1, 2, 3, 4.

Example: Consider the following transportation problem (Table 3.3) and develop a linear programming (LP) model.

Table 5.3: Transportation Problem

Source	Destination			
	1	**2**	**3**	**Supply**
1	15	20	30	350
2	10	9	15	200
3	14	12	18	400
Demand	250	400	300	

Solution:

Let x_{ij} be the number of units to be transported from the source i to the destination j, where i = 1, 2, 3,…m and j = 1, 2, 3,…n.

The linear programming model is

Minimize $Z = 15x_{11}+20x_{12}+30x_{13}+10x_{21}+9x_{22}+15x_{23}+14x_{31}+12x_{32}+18x_{33}$

Subject to constraints,

$x_{11}+x_{12}+x_{13} \leq 350$ (i)

$x_{21}+x_{22}+x_{23} \leq 200$ (ii)

$x_{31}+x_{32}+x_{33} \leq 400$ (iii)

$x_{11}+x_{12}+x_{31} = 250$ (iv)

$x_{12}+x_{22}+x_{32} = 400$ (v)

$x_{13}+x_{23}+x_{33} = 300$ (vi)

$x_{ij} \geq 0$ for all i and j.

Notes

In the above LP problem, there are m x n = 3 x 3 = 9 decision variables and m + n = 3 + 3 = 6 constraints.

5.4 Maximization Transportation Problem

In this type of problem, the objective is to maximize the total profit or return. In this case, convert the maximization problem into minimization by subtracting all the unit cost from the highest unit cost given in the table and solve.

Example: A manufacturing company has four plants situated at different locations, all producing the same product. The manufacturing cost varies at each plant due to internal and external factors. The size of each plant varies, and hence the production capacities also vary. The cost and capacities at different locations are given in the following table:

Table 5.4: Cost and Capacity of Different Plants

Particulars	Plant			
	A	B	C	D
Production cost per unit (Rs.)	18	17	15	12
Capacity	150	250	100	70

The company has five warehouses. The demands at these warehouses and the transportation costs per unit are given in the Table 5.5 below. The selling price per unit is ₹ 30.

Table 5.5: Transportation Problem

Warehouse	Transportation cost (Rs) — Unit-wise				Demand
	A	B	C	D	
1	6	9	5	3	100
2	8	10	7	7	200
3	2	6	3	8	120
4	11	6	2	9	80
5	3	4	8	10	70

1. Formulate the problem to maximize profits.
2. Find the total profit.

Solution:

1. The objective is to maximize the profits. Formulation of transportation problem as profit matrix table is shown in Table 5.6. The profit values are arrived as follows.

   Profit = Selling Price –Production cost –Transportation cost

Notes

Table 5.6: Profit Matrix

	Destination				
	A	**B**	**C**	**D**	**Demand**
1	6	4	10	15	100
2	4	3	8	11	200
3	10	7	12	10	120
4	1	7	13	9	80
5	9	9	7	8	70
Supply	150	250	100	70	570

Converting the profit matrix to an equivalent loss matrix by subtracting all the profit values in Table 5.6 from the highest value 13. Subtracting all the values from 13, the loss matrix obtained is shown in the Table 5.7.

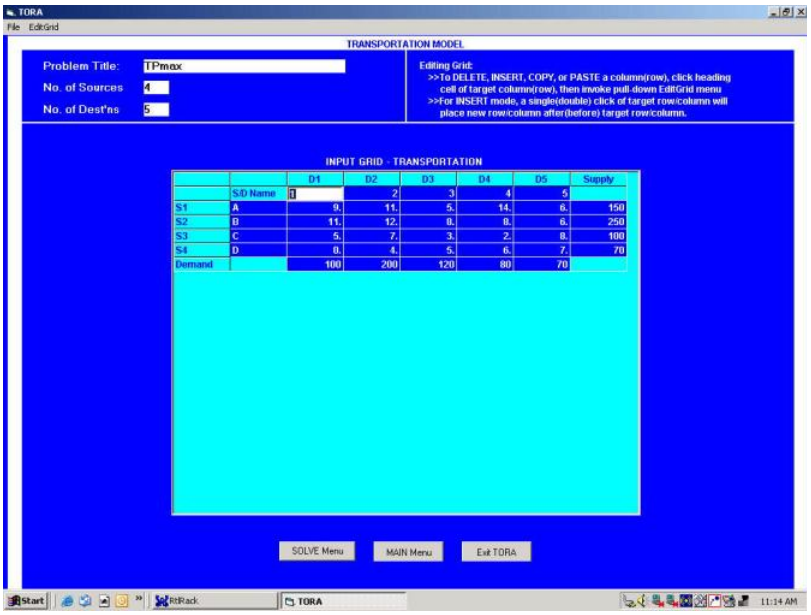
Table 5.7: Loss Matrix

	Destination				
	A	**B**	**C**	**D**	**Demand**
1	9	11	5	0	100
2	11	12	7	4	200
3	5	8	3	5	120
4	14	8	2	6	80
5	6	6	8	7	70
Supply	150	250	100	70	570

2. To determine the initial solution using TORA

Input Screen:

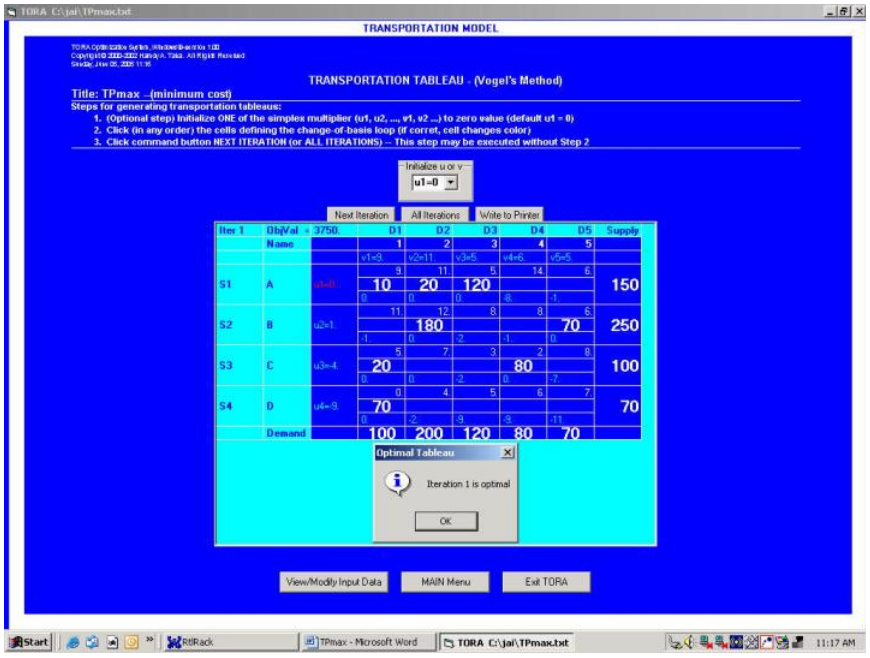
Table 5.8 (a): TORA, Input Screen for TP Max Problem



Output Screen:

Notes

Table 5.8 (b): TORA Output Screen (Vogel's Method)



The first iteration itself is optimal, hence optimality is reached.

3. To find the total cost:

The total maximization profit associated with the solution is

Total Profit = (6 × 10) + (4 × 20) + (10 × 120) + (3 × 180) + (9 × 70) + (10 × 20) + (13 × 80) + (15 × 70)

= 60 + 80 + 1200 + 540 + 630 + 200 + 1040 + 1050

= ₹ 4800.00

5.5 Balanced Transportation Problem

When the total supplies of all the sources are equal to the total demand of all destinations, the problem is a ***balanced transportation problem.***

Total supply = Total demand

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

The problem given in Example 3.1 represents a balanced transportation problem.

5.6 Unbalanced Transportation Problem

When the total supply of all the sources is not equal to the total demand of all destinations, the problem is an ***unbalanced transportation problem.***

Notes

Total supply ≠ Total demand

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

5.6.1 Demand Less than Supply

In real-life, supply and demand requirements will rarely be equal. This is because of variation in production from the supplier end, and variations in forecast from the customer end. Supply variations may be because of shortage of raw materials, labour problems, improper planning and scheduling. Demand variations may be because of change in customer preference, change in prices and introduction of new products by competitors. These unbalanced problems can be easily solved by introducing dummy sources and dummy destinations. If the total supply is greater than the total demand, a dummy destination (dummy column) with demand equal to the supply surplus is added. If the total demand is greater than the total supply, a dummy source (dummy row) with supply equal to the demand surplus is added. The unit transportation cost for the dummy column and dummy row are assigned zero values, because no shipment is actually made in case of a dummy source and dummy destination.

Example:

Check whether the given transportation problem shown in Table 5.9 is a balanced one. If not, convert the unbalanced problem into a balanced transportation problem.

Table 5.9: Transportation Model with Supply exceeding Demand

Source	Destination			Supply
	1	2	3	
1	25	45	10	200
2	30	65	15	100
3	15	40	55	400
Demand	200	100	300	

Solution:

For the given problem, the total supply is not equal to the total demand.

$$\sum_{i=1}^{3} a_i \neq \sum_{j=1}^{3} b_j$$

since,

$$\sum_{i=1}^{3} a_i = 700 \text{ and } \sum_{j=1}^{3} b_j = 600$$

The given problem is an unbalanced transportation problem. To convert the unbalanced transportation problem into a balanced problem, add a dummy destination (dummy column).

i.e., the demand of the dummy destination is equal to,

$$\sum_{i=1}^{3} a_i - \sum_{j=1}^{3} b_j$$

Notes

Thus, a dummy destination is added to the table, with a demand of 100 units. The modified table is shown in Table 5.10 which has been converted into a balanced transportation table. The unit costs of transportation of dummy destinations are assigned as zero.

Table 5.10: Dummy Destination Added

Source	Destination				Supply
	1	2	3	4	
1	25	45	10	0	200
2	30	65	15	0	100
3	15	40	55	0	400
Demand	200	100	300	100	700/700

Similarly,

If $\sum_{j=1}^{n} b_j > \sum_{i=1}^{m} a_i$ then include a dummy source to supply the excess demand.

5.6.2 Demand Greater than Supply

Example: Convert the transportation problem shown in Table 3.6 into a balanced problem.

Table 5.11: Demand Exceeding Supply

Source	Destination				Supply
	1	2	3	4	
1	10	16	9	12	200
2	12	12	13	5	300
3	14	8	13	4	300
Demand	100	200	450	250	1000/800

Solution:

The given problem is, $\sum_{j=1}^{4} b_j > \sum_{i=1}^{3} a_j$

$\sum_{i=1}^{3} a_i = 800$ and $\sum_{j=1}^{4} b_j = 1000$

The given problem is an unbalanced one. To convert it into a balanced transportation problem, include a dummy source (dummy row) as shown in Table 5.12

Notes

Table 5.12: Balanced TP Model

Source	Destination				
	1	2	3	4	Supply
1	10	16	9	12	200
2	12	12	13	5	300
3	14	8	13	4	300
4	0	0	0	0	200
Demand	100	200	450	250	1000/1000

5.7 Initial Feasible Solution

Step 1: Formulate the Problem

Formulate the given problem and set up in a matrix form. Check whether the problem is a balanced or unbalanced transportation problem. If unbalanced, add dummy source (row) or dummy destination (column) as required.

Step 2: Obtain the Initial Feasible Solution

The initial feasible solution can be obtained by any of the following three methods.

1. Northwest Corner Method (NWC)
2. Row and Column Minima Method (RCMM)
3. Vogel's Approximation Method (VAM)

The transportation cost of the initial basic feasible solution through Vogel's approximation method, VAM will be the least when compared to the other two methods which gives the value nearer to the optimal solution or optimal solution itself. Algorithms for all the three methods to find the initial basic feasible solution are given.

5.7.1 Algorithm for North-West Corner Method (NWC)

1. Select the North-west (i.e., upper left) corner cell of the table and allocate the maximum possible units between the supply and demand requirements. During allocation, the transportation cost is completely discarded (not taken into consideration).
2. Delete that row or column which has no values (fully exhausted) for supply or demand.
3. Now, with the new reduced table, again select the North-west corner cell and allocate the available values.
4. Repeat steps (2) and (3) until all the supply and demand values are zero.
5. Obtain the initial basic feasible solution.

Notes

Example: Find out the initial feasible solution for transportation cost involved in the problem shown through Table 5.13.

Table 5.13

	Retail shops				
Factories	**1**	**2**	**3**	**4**	**Supply**
1	3	5	7	6	50
2	2	5	8	2	75
3	3	6	9	2	25
Demand	20	20	50	60	

Solution:

Table 5.14

	Retail shops				
Factories	**1**	**2**	**3**	**4**	**Supply**
1	3^{20}	5^{20}	7^{10}	6	~~50~~
2	2	5	8^{40}	2^{35}	~~75~~
3	3	6	9	2^{25}	~~25~~
Demand	~~20~~	~~20~~	~~50~~	~~60~~	

As under the process of NWC method, we allocate $x_{11} = 20$. Now demand for the first column is satisfied, therefore, eliminate that column.

Proceeding in this way, we observe that

$$x_{12} = 20, x_{13} = 10, x_{23} = 40, x_{24} = 35, x_{34} = 25.$$

Delete the row if supply is exhausted.

Delete the column if demand is satisfied.

Here, number of retail shops (n) = 4, and

Number of factories (m) = 3

Number of basic variables = m + n – 1 = 3 + 4 – 1 = 6.

Initial basic feasible solution:

$$20 \times 3 + 20 \times 5 + 10 \times 7 + 40 \times 8 + 35 \times 2 + 25 \times 2 = 670$$

5.7.2 Algorithm for Row and Column Minima Method or Least Cost Method (LCM)

1. Select the smallest transportation cost cell available in the entire table and allocate the supply and demand.
2. Delete that row/column which has exhausted. The deleted row/column must not be considered for further allocation.
3. Again select the smallest cost cell in the existing table and allocate. (*Note:* In case, if there are more than one smallest costs, select the cells where maximum allocation can be made)
4. Obtain the initial basic feasible solution.

Notes

Example: Solve example 5 by least cost method.

Solution:

To make our understanding better, let us go through the Table 5.15.

Table 5.15

	Retail shops				
Factories	**1**	**2**	**3**	**4**	**Supply**
1	3	5	7	6	50
2	2	5	8	2	75
3	3	6	9	2	25
Demand	20	20	50	60	

Applying the least cost method-

Table 5.16

	Retail shops				
Factories	**1**	**2**	**3**	**4**	**Supply**
1	3	5^{20}	7^{30}	6	~~50~~
2	2^{20}	5	8	2^{55}	~~75~~
3	3	6	9^{20}	2^{5}	~~25~~
Demand	~~20~~	~~20~~	~~50~~	~~60~~	

We observe that $c_{21} = 2$, which is the minimum transportation cost. So, $x_{21} = 20$.

Proceeding in this way, we observe that $x_{24} = 55$, $x_{34} = 5$, $x_{12} = 20$, $x_{13} = 30$, $x_{33} = 20$.

Number of basic variables = m + n –1 = 3 + 4 – 1 = 6.

The initial basic feasible solution:

$= 20 \times 2 + 55 \times 2 + 5 \times 2 + 20 \times 5 + 30 \times 7 + 20 \times 9$

$= 650.$

5.7.3 Algorithm for Vogel's Approximation Method (VAM)

1. Calculate penalties for each row and column by taking the difference between the smallest cost and next highest cost available in that row/column. If there are two smallest costs, then the penalty is zero.
2. Select the row/column, which has the largest penalty and make allocation in the cell having the least cost in the selected row/column. If two or more equal penalties exist, select one where a row/column contains minimum unit cost. If there is again a tie, select one where maximum allocation can be made.
3. Delete the row/column, which has satisfied the supply and demand.
4. Repeat steps (1) and (2) until the entire supply and demands are satisfied.
5. Obtain the initial basic feasible solution.

Notes

Caution The initial solution obtained by any of the three methods must satisfy the following conditions.

1. The solution must be feasible, i.e., the supply and demand constraints must be satisfied (also known as rim conditions).
2. The number of positive allocations, N must be equal to m + n – 1, where m is the number of rows and n is the number of columns.

Example: Find the initial feasible solution of the transportation problem illustrated through Table 5.17

Table 5.17

	Destination				
Origin	**1**	**2**	**3**	**4**	**Supply**
1	20	22	17	4	120
2	24	37	9	7	75
3	34	37	20	15	25
Demand	60	40	30	110	240

Solution:

Solving the problem through Vogel's Approximation Method, we get the Table 5.18

Table 5.18

	Destination					
Origin	**1**	**2**	**3**	**4**	**Supply**	**Penalty**
1	20	22^{40}	17	4	~~120~~ 80	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
Demand	60	~~40~~	30	110	240	
Penalty	4	15	8	3		

The highest penalty occurs in the second column. The minimum c_{ij} in this column is c_{12} (i.e., 22). Hence, x_{12} = 40 and the second column is eliminated.

Now again calculate the penalty.

Table 5.19

Origin	**1**	**2**	**3**	**4**	**Supply**	**Penalty**
1	20	22^{40}	17	4^{80}	~~120~~	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
Demand	60	~~40~~	30	110	240	
Penalty	4		8	3		

The highest penalty occurs in the first row. The minimum cij in this row is c_{14} (i.e., 4). So x_{14} = 80 and the first row is eliminated.

Notes

Final table:

Now assuming that you can calculate the values yourself, we reach the final table as in Table 5.20

Table 5.20

Destination											
Origin	**1**	**2**	**3**	**4**	**Supply**	**Penalty**					
1	20	**22** 40	17	**4** 80	~~120~~	3	13	-	-	-	-
2	**24** 10	37	**9** 30	**7** 30	~~70~~	2	2	2	17	24	24
3	**32** 50	37	20	15	~~50~~	5	5	5	17	32	-
Demand	~~60~~	~~40~~	~~30~~	~~110~~	240						
Penalty	4	15	8	3							
	4	-	8	3							
	8	-	11	8							
	8	-	-	8							
	8	-	-	-							
	24	-	-	-							

The initial basic feasible solution:

$= 22 \times 40 + 4 \times 80 + 24 \times 10 + 9 \times 30 + 7 \times 30 + 32 \times 50$

$= 3520$

Self Assessment

State true or false:

1. Transportation problem applies to situations where a set of commodities is to be transported from source to another.
2. The cost of transportation per unit varies between different supply points,
3. In transportation problems, there are supply constraints for each destination.
4. The most important objective of a transportation problem is to maximize the cost of shipping.
5. The transportation problem is an extension of transshipment problem.
6. There may be routes that are unavailable to transport units from one source to one or more destinations.
7. In transshipment problem, each node makes supplies to the other.
8. Degeneracy is the condition when the number of filled cells is less than the number of rows plus the number of columns minus one.

5.8 Degeneracy in Transportation Problems

Notes

Degeneracy involves two steps:

1. ***Check for degeneracy:*** The solution that satisfies the above said conditions N = m + n – 1 is a non-degenerate basic feasible solution otherwise, it is a degenerate solution. Degeneracy may occur either at the initial stage or at subsequent iterations.

   If number of allocations, N = m + n – 1, then degeneracy does not exist, one has to go to the next step.

   If number of allocations, N = m + n – 1, then degeneracy does exist, and has to be resolved before going to the next step.

2. ***Resolving Degeneracy:*** To resolve degeneracy at the initial solution, allocate a small positive quantity e to one or more unoccupied cell that have lowest transportation costs, so as to make m + n – 1 allocations (i.e., to satisfy the condition N = m + n – 1). The cell chosen for allocating e must be of an independent position. In other words, the allocation of e should avoid a closed loop and should not have a path.

The following Table 5.21 shows independent allocations.

Table 5.21: Independent Allocations

*	*	*
	*	

*	*		
		*	
	*		*

The following Tables 5.22 (a), (b) and (c) show non-independent allocations.

Table 5.22: (a) Independent Allocations, (b) and (c)

	*	*
	*	*

(a)

	*		*
	*		*

(b)

*	*		
	*		
*	*		

(c)

Notes

Self Assessment

Fill in the blanks:

9. Degeneracy involves …….steps.
10. To resolve degeneracy at the initial solution, allocate a small positive quantity e to one or more unoccupied cell that have ……..transportation costs.
11. ……..may occur either at the initial stage or at subsequent iterations.

Case Study

South India Soaps Ltd.

The South India Soaps Ltd. (SISOL) operated 3 factories from which it shipped soaps to regional warehouses. In 2007, the demand for soaps was 24,000 tonnes distributed as follows:

Region	Demand in 000′ tones
Cochin	5
Nellore	4
Salem	4
Madurai	11
	24

One-shift production capacity in each of the 3 factories was as follows:

Region	Demand in 000′ tones
Chennai	12
Coimbatore	7
Bangalore	7
	26

Estimated transport costs (in hundred rupees per 000′ tones) are given below:

Factory	Regional warehouses			
	Cochin	Nellore	Salem	Madurai
Chennai	95	105	80	15
Coimbatore	115	180	40	30
Bangalore	155	180	95	70

SISOL followed a policy of decentralization under which each of the four regional warehouses was under the direct supervision of a regional sales manager and he was responsible for the profitability of operation under his control.

Over a period of time, this procedure lead to increasing friction in the organization. There were questions whether this procedure achieved the objectives of minimizing transport costs; also there was no coordination. For instance in 2005, the sales manager of Madurai & Nellore placed their orders with the Chennai factory which did not have the capacity to meet all demands. This led to inefficient and duplicate orders, friction etc. The final pattern that emerged in 2005 was as follows:

Contd...

Notes

	Cochin	Nellore	Salem	Madurai
Chennai	0	1	2	9
Coimbatore	3	0	0	2
Bangalore	2	3	2	0

The general manager of SISOL called meeting of the executives at the central office. Some executives proposed that all orders should be routed through the central office which would determine the optimal programme. Others protested that this would seriously conflict with the firm's philosophy of decentralization.

Question:

You have been hired as a consultant by the general manager. Prepare a minimum cost distribution schedule for SISOL. Compare this schedule with the present schedule of 2005; which is better?

5.9 Summary

- A transportation problem basically deals with that problem which aims to find the best way to fulfill the demand of various demand sources using the capacities of various supply points.
- Although the formation can be used to represent more general assignment and scheduling problems as well as transportation and distribution problems, it gets its name from its application to problems involving transporting products from several sources to several destinations.
- The two common objectives of such problems are either to minimize the cost of shipping commodity to various destinations or to maximize the profit of shipping it various destinations.
- Transportation problems are often used in, surprise, transportation planning. But it should always be remembered that the transportation problem is only a special topic of the linear programming problems.

5.10 Keywords

Balanced Transportation Problem: when the total supplies of all the sources are equal to the total demand of all destinations.

Basic Feasible Solution: A feasible solution to an 'm' origin and 'n; destination is said to be basic, if the number of positive allocations are (m+n+1).

Degeneracy: When the number of filled cells is less than the number of rows plus the number of columns minus one.

Notes

5.11 Review Questions

1. Find the initial basic feasible solution for the transportation problem given in following table.

From	To			Available
	A	B	C	
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Requirement	4	2	2	

2. Obtain an optimal solution for the transportation problem by MODI method given in this table:

		Destination				
		D_1	D_2	D_3	D_4	Supply
Source	S_1	19	30	50	10	7
	S_2	70	30	40	60	9
	S_3	40	8	70	20	18
	Demand	5	8	7	14	

3. Solve the transportation problem

		Destination			
		1	2	3	Supply
Source	1	3	5	7	10
	2	11	8	9	8
	3	13	3	9	5
Demand		5	9	11	23 / 25

4. Develop a network representation of the transportation problem for a company that manufactures products at three plants and ships them to three warehouses. The plant capacities and warehouse demands are shown in the following table:

The transportations cost per unit (in ₹) is given in matrix.

Plant	Warehouse			Plant Capacity (no. of units)
	W1	W2	W3	
P1	22	18	26	350
P2	12	12	10	450
P3	14	20	10	200
Warehouse demand (no. of units)	250	450	300	

5. Determine whether a dummy source or a dummy destination is required to balance the model given.

   (a) Supply $a_1 = 15$, $a_2 = 5$, $a_3 = 4$, $a_4 = 6$

   Demand $b_1 = 4$, $b_2 = 15$, $b_3 = 6$, $b_4 = 10$

Notes

(b) Supply $a_1 = 27$, $a_2 = 13$, $a_3 = 10$

Demand $b_1 = 30$, $b_2 = 10$, $b_3 = 6$, $b_4 = 10$

(c) Supply $a_1 = 2$, $a_2 = 3$, $a_3 = 5$

Demand $b_1 = 3$, $b_2 = 2$, $b_3 = 2$, $b_4 = 2$, $b_5 = 1$.

6. A state has three power plants with generating capacities of 30, 40 and 25 million KWH that supply electricity to three cities located in the same state. The demand requirements (maximum) of the three cities are 35, 40 and 20 million KWH. The distribution cost (₹ in thousand) per million unit for the three cities are given in the table below:

City

Plant		1	2	3
	1	60	75	45
	2	35	35	40
	3	55	50	45

(a) Formulate the problem as a transportation model.

(b) Determine an economical distribution plan.

(c) If the demand is estimated to increase by 15%, what is your revised plan?

(d) If the transmission loss of 5% is considered, determine the optimal plan.

7. Find the initial transportation cost for the transportation matrix given using North-West corner method, Least cost method and Vogel's Approximation method.

Destination

		1	2	3	4	**Supply**
	A	5	6	7	8	25
Source	**B**	7	5	4	2	75
	C	6	1	3	2	15
Demand		50	30	20	15	

8. The Sharp Manufacturing Company produces three types of monoblock pumps for domestic use. Five machines are used for manufacturing the pumps. The production rate varies for each machine and also the unit product cost. Daily demand and machine availability are given below.

Demand Information

Product			
	A	**B**	**C**
Demand (units)	2000	15000	700

Machine Availability Details

Machine capacity (units)					
	1	**2**	**3**	**4**	**5**
Available	700	1000	1500	1200	800

Notes

Unit Product Cost

	Product		
Machine	**A**	**B**	**C**
1	150	80	75
2	120	95	60
3	112	100	60
4	121	95	50
5	125	75	50

Determine the minimum production schedule for the products and machines.

9. A company has plants at locations A, B and C with the daily capacity to produce chemicals to a maximum of 3000 kg, 1000 kg and 2000 kg respectively. The cost of production (per kg) are ₹ 800, ₹ 900 and ₹ 7.50 respectively. Customer's requirement of chemicals per day is as follows:

Customer	Chemical Required	Price offered
1	2000	200
2	1000	215
3	2500	225
4	1000	200

Transportation cost (in rupees) per kg from plant locations to customer's place is given in table.

		Customer			
		1	2	3	4
	A	5	7	10	12
Plant	B	7	3	4	2
	C	4	6	3	9

Find the transportation schedule that minimizes the total transportation cost.

10. A transportation model has four supplies and five destinations. The following table shows the cost of shipping one unit from a particular supply to a particular destination.

Source	Destination					Supply
	1	**2**	**3**	**4**	**5**	
1	13	6	9	6	10	13
2	8	2	7	7	9	15
3	2	12	5	8	7	13
Demand	10	15	7	10	2	

The following feasible transportation pattern is proposed:

$x_{11} = 10$, $x_{12} = 3$, $x_{22} = 9$, $x_{23} = 6$, $x_{33} = 9$, $x_{34} = 4$, $x_{44} = 9$, $x_{45} = 5$.

Test whether these allocations involve least transportation cost. If not, determine the optimal solution.

Notes

11. A company has four factories situated in four different locations in the state and four company showrooms in four other locations outside the state. The per unit sale price, transportation cost and cost of production is given in table below, along with weekly requirement.

Factory	Showrooms				Cost of production (₹)
	1	**2**	**3**	**4**	
A	9	4	5	3	12
B	4	4	4	4	17
C	4	6	5	6	19
D	8	7	7	4	17

Factory	Weekly Capacity (units)	Weekly demand (units)
A	15	10
B	20	14
C	25	20
D	20	22

Determine the weekly distribution schedule to maximize the sales profits.

12. A computer manufacturer has decided to launch an advertising campaign on television, magazines and radio. It is estimated that maximum exposure for these media will be 70, 50, and 40 million respectively. According to a market survey, it was found that the minimum desired exposures within age groups 15-20, 21-25, 26-30, 31-35 and above 35 are 10, 20, 25, 35 and 55 million respectively. The table below gives the estimated cost in paise per exposure for each of the media. Determine an advertising plan to minimize the cost.

Media	Age Groups				
	15-20	**21-25**	**26-30**	**31-35**	**above 35**
TV	14	9	11	11	12
Radio	11	7	6	7	8
Magazine	9	10	7	10	8

Solve the problem and find the optimal solution, i.e., maximum coverage at minimum cost.

13. A garment manufacturer has 4 units I, II, III, and IV, the production from which are received by 4 direct customers. The weekly production of each manufacturing unit is 1200 units and all the units are of the same capacity. The company supplies the entire production from one unit to one supplier. Since the customers are situated at different locations, the transportation cost per unit varies. The unit cost of transportation is given in the table. As per the company's policy, the supply from unit B is restricted to customer 2 and 4, and from unit D to customer 1 and 3. Solve the problem to cope with the supply and demand constraints.

Manufacturing unit	1	2	3	4
A	4	6	8	3
B	4	–	5	–
C	6	5	5	9
D	–	7	–	6

Notes

14. A company dealing in home appliances has a sales force of 20 men who operate from three distribution centers. The sales manager feels that 5 salesmen are needed to distribute product line I, 6 to distribute product line II, 5 for product line III and 4 to distribute product line IV. The cost (in ₹) per day of assigning salesmen from each of the offices are as follows:

		Product Line			
		I	**II**	**III**	**IV**
Source	A	10	12	13	9
	B	9	11	12	13
	C	7	8	9	10

Currently, 8 salesmen are available at center A, 5 at center B and 7 at center C. How many salesmen should be assigned from each center to sell each product line, in order to minimize the cost? Is the solution unique?

15. Solve the following degenerate transportation problem:

		Destination			
		I	**II**	**III**	**Supply**
Source	A	7	3	4	2
	B	2	1	3	3
	C	3	4	6	5
	Demand	4	1	5	

Answers: Self Assessment

1. False
2. False
3. False
4. False
5. False
6. True
7. False
8. True
9. Two
10. Lowest
11. Degeneracy

5.12 Further Readings

Books

G Dantzig, *Linear Programming and Extensions,* Princeton University Press.

G Dantzig, M N Thapa, *Linear Programming 2: Theory and Extensions,* Princeton University Press.

Robert J. Vanderbei, *Linear Programming: Foundations and Extensions,* Princeton University Press.

S Jaisanker, *Quantitative Techniques for Management,* Excel Books.

Online links

www.adbook.net

www.mathresources.com

Notes

Unit 6: Transportation Problem – Optimality Tests

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Objectives

After studying this unit, you will be able to:

- Understand the significance of optimality tests in transportation problem
- Learn to drive optimal solution using Modified Distribution method and Stepping stone Method
- Construct the transshipment transportation table for transshipment problem
- Examine multiple optimal solutions, and prohibited routes in the transportation problem

Introduction

Once the initial feasible solution is reached, the next step is to check the optimality. An optimal solution is one where there is no other set of transportation routes that would reduce the total transportation cost, for which we have to evaluate each unoccupied cell (which represents unused routes) in terms of opportunity cost. In this process, if there is no negative opportunity cost, the solution is an optimal solution.

6.1 Techniques of Finding Optimal Solution

Optimality can be tested by two ways namely:

1. Stepping Stone Method
2. Modified Distribution Method

Let us understand each of them one by one.

Notes

6.2 Stepping Stone Method

It is a method for computing optimum solution of a transportation problem.

Steps Involved:

Step 1: Determine an initial basic feasible solution using any one of the following:

(a) North West Corner Rule

(b) Matrix Minimum Method

(c) Vogel Approximation Method

Step 2: Make sure that the number of occupied cells is exactly equal to m+n–1, where m is the number of rows and n is the number of columns.

Step 3: Select an unoccupied cell.

Step 4: Beginning at this cell, trace a closed path using the most direct route through at least three occupied cells used in a solution and then back to the original occupied cell and moving with only horizontal and vertical moves. The cells at the turning points are called "Stepping Stones" on the path.

Step 5: Assign plus (+) and minus (-) signs alternatively on each corner cell of the closed path just traced, starting with the plus sign at unoccupied cell to be evaluated.

Step 6: Compute the net change in the cost along the closed path by adding together the unit cost figures found in each cell containing a plus sign and then subtracting the unit costs in each square containing the minus sign.

Step 7: Check the sign of each of the net changes. If all the net changes computed are greater than or equal to zero, an optimum solution has been reached. If not, it is possible to improve the current solution and decrease the total transportation cost.

Step 8: Select the unoccupied cell having the most negative net cost change and determine the maximum number of units that can be assigned to a cell marked with a minus sign on the closed path corresponding to this cell. Add this number to the unoccupied cell and to all other cells on the path marked with a plus sign. Subtract this number from cells on the closed path marked with a minus sign.

Step 9: Repeat the procedure until you get an optimum solution

Example: Consider the following transportation problem (cost in rupees). Find the optimum solution

Factory	**D**	**E**	**F**	**G**	**Capacity**
A	4	6	8	6	700
B	3	5	2	5	400
C	3	9	6	5	600
Requirement	400	450	350	500	1700

Notes

Solution: First, we find out an initial basic feasible solution by Matrix Minimum Method

Factory	**D**	**E**	**F**	**G**	**Capacity**
A	4	6^{450}	8	6^{250}	700
B	3^{50}	5	2^{350}	5	400
C	3^{350}	9	6	5^{250}	600
Requirement	400	450	350	500	1700

Here, m + n – 1 = 6. So the solution is not degenerate.

The cell AD (4) is empty so allocate one unit to it. Now draw a closed path from AD.

Factory	**D**	**E**	**F**	**G**	**Capacity**
A	4^{+1}	6^{450}	8	6^{249}	700
B	3^{50}	5	2^{350}	5	400
C	3^{349}	9	6	5^{251}	600
Requirement	400	450	350	500	1700

The increase in the transportation cost per unit quantity of reallocation is + 4 – 6 + 5 – 3 = 0.

This indicates that every unit allocated to route AD will neither increase nor decrease the transportation cost. Thus, such a reallocation is unnecessary.

Choose another unoccupied cell. The cell BE is empty so allocate one unit to it.

Factory	**D**	**E**	**F**	**G**	**Capacity**
A	4	6^{449}	8	6^{251}	700
B	3^{49}	5^{+1}	2^{350}	5	400
C	3^{351}	9	6	5^{249}	600
Requirement	400	450	350	500	1700

The increase in the transportation cost per unit quantity of reallocation is + 5 – 6 + 6 – 5 + 3 – 3 = 0

This indicates that every unit allocated to route BE will neither increase nor decrease the transportation cost. Thus, such a reallocation is unnecessary.

The allocations for other unoccupied cells are:

Unoccupied cells	**Increase in cost per unit of reallocation**	**Remarks**
CE	+ 9 – 6 + 6 – 5 = 4	Cost Increases
CF	+ 6 – 3 + 3 – 2 = 4	Cost Increases
AF	+ 8 – 6 + 5 – 3 + 3 – 2 = 5	Cost Increases
BG	+ 5 – 5 + 3 – 3 = 0	Neither increase nor decrease

Since all the values of unoccupied cells are greater than or equal to zero, the solution obtained is optimum.

Minimum transportation cost is:

$$6 \times 450 + 6 \times 250 + 3 \times 250 + 2 \times 250 + 3 \times 350 + 5 \times 250 = ₹\ 7350$$

Notes

Self Assessment

Multiple Choice Questions:

1. When total supply is equal to the total demand in a transportation problem, the problem is said to be

   (a) Balanced (b) Unbalanced

   (c) Degenerate (d) All of the above

2. Which of the following methods is used to verify the optimality of the current solution of the transportation problem?

   (a) Least Cost Method (b) Vogel's Approximation Method

   (c) MODI method (d) All of the above

3. MODI method and stepping stone method is used for this purpose

   (a) To find out initial basic feasible solution

   (b) Optimal solution

   (c) Multiple solution

   (d) All of the above

6.3 Modified Distribution Method (MODI)

Another method to check optimality is that of MODI.

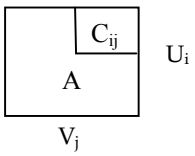
Steps Involved:

Step 1: Row 1, row 2,…, row i of the cost matrix are assigned with variables U_1, U_2, …, U_i and the column 1, column 2,…, column j are assigned with variables V_1, V_2, …,V_j respectively.

Step 2: Initially, assume any one of U_i values as zero and compute the values for U_1, U_2, …,U_i and V_1, V_2, …,V_j by applying the formula for occupied cell.

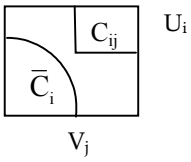
For occupied cells,

$C_{ij} + U_i + V_j = 0$



Step 3: Obtain all the values of C_{ij} for unoccupied cells by applying the formula for unoccupied cell. For unoccupied cells,

Opportunity Cost, $= C_{ij} + U_i + V_j$



Notes

If $\bar{C}_{ij}$ values are > 0 then, the basic initial feasible solution is optimal.

If $\bar{C}_{ij}$ values are = 0 then, the multiple basic initial feasible solution exists.

If $\bar{C}_{ij}$ values are < 0 then, the basic initial feasible solution is not optimal.

Notes The MODI method is based on the concept of Duality.

Did u know? MODI method is also known as u-v method or method of multipliers.

6.4 Procedure for Shifting of Allocations

Select the cell which has the most negative $\bar{C}_{ij}$ value and introduce a positive quantity called 'θ' in that cell. To balance that row, allocate a '– θ' to that row in occupied cell. Again, to balance that column put a positive 'θ' in an occupied cell and similarly a '–θ' to that row. Connecting all the 'θ's and '–θ's, a closed loop is formed.

Two cases are represented in Table 6.1 (a) and 6.1 (b). In Table 6.1 (a) if all the θ allocations are joined by horizontal and vertical lines, a closed loop is obtained.

The set of cells forming a closed loop is

CL = {(A, 1), (A, 3), (C, 3), (C, 4), (E, 4), (E, 1), (A, 1)}

The loop in Table 4.22(b) is not allowed because the cell (D3) appears twice.

Table 6.1(a)

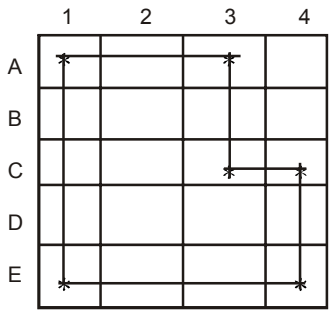
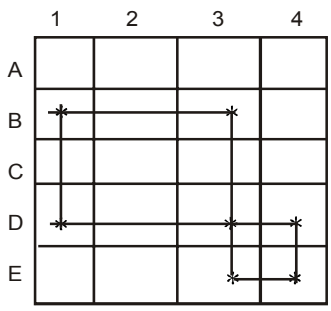


Table 6.1(b)



Notes

Conditions for Forming a Loop

1. The start and end points of a loop must be the same.
2. The lines connecting the cells must be horizontal and vertical.
3. The turns must be taken at occupied cells only.
4. Take a shortest path possible (for easy calculations).

Notes

1. Every loop has an even number of cells and at least four cells.
2. Each row or column should have only one '+' and '-' sign.
3. Closed loop may or may not be square in shape. It can also be a rectangle or a stepped shape.
4. It doesn't matter whether the loop is traced in a clockwise or anti-clockwise direction.

Take the most negative '– q' value, and shift the allocated cells accordingly by adding the value in positive cells and subtracting it in the negative cells. This gives a new improved table. Then go to step 5 to test for optimality.

Calculate the Total Transportation Cost

Since all the $\bar{C}_{ij}$ values are positive, optimality is reached and hence the present allocations are the optimum allocations. Calculate the total transportation cost by summing the product of allocated units and unit costs.

Task Find the initial basic solution for the transportation problem and hence solve it.

Source	Destination				Supply
	1	2	3	4	
1	4	2	7	3	250
2	3	7	5	8	450
3	9	4	3	1	500
Demand	200	400	300	300	**1200**

6.5 Prohibited Routes Problem

In practice, there may be routes that are unavailable to transport units from one source to one or more destinations. The problem is said to have an unacceptable or prohibited route. To overcome such kind of transportation problems, assign a very high cost to prohibited routes, thus preventing them from being used in the optimal solution regarding allocation of units.

6.6 Transshipment Problem

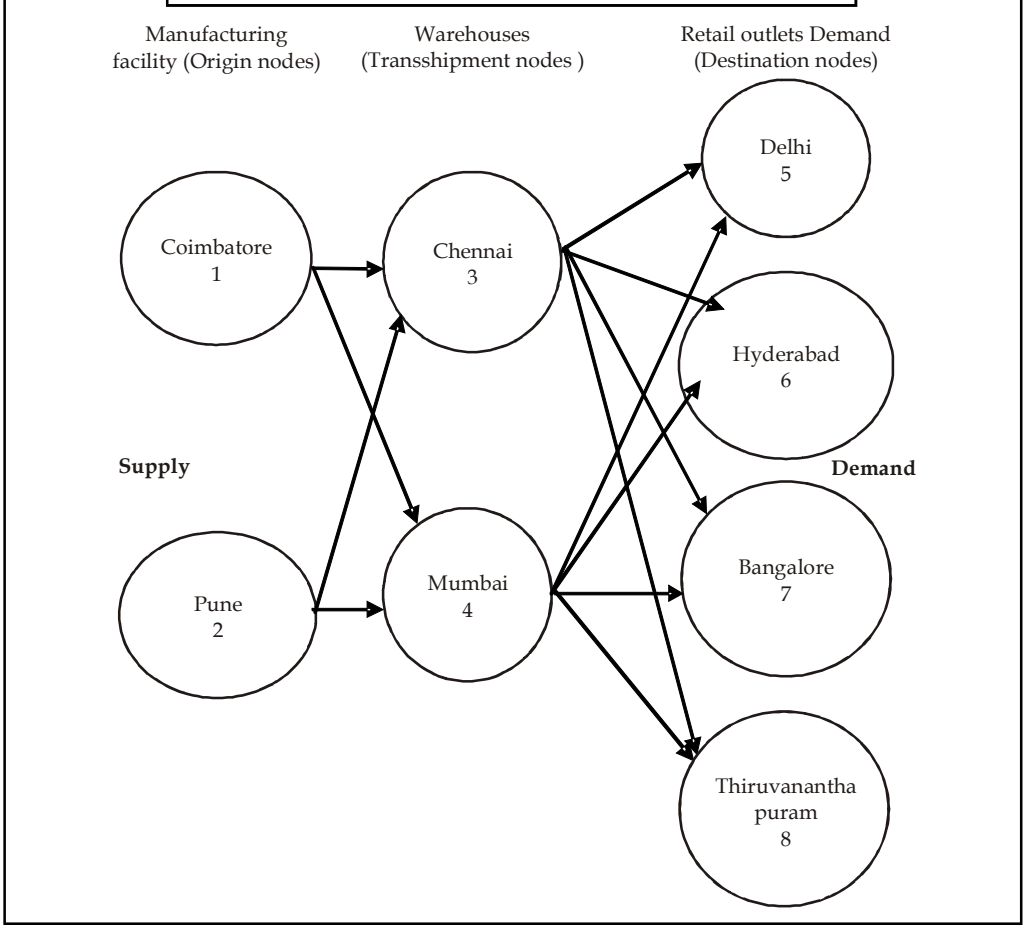
The transshipment problem is an extension of the transportation problem in which the commodity can be transported to a particular destination through one or more intermediate or transshipment nodes.

Notes

Each of these nodes in turn supply to other destinations. The objective of the transshipment problem is to determine how many units should be shipped over each node so that all the demand requirements are met with the minimum transportation cost.

Considering a company with its manufacturing facilities situated at two places, Coimbatore and Pune. The units produced at each facility are shipped to either of the company's warehouse hubs located at Chennai and Mumbai. The company has its own retail outlets in Delhi, Hyderabad, Bangalore and Thiruvananthapuram. The network diagram representing the nodes and transportation per unit cost is shown in Figure 6.1. The supply and demand requirements are also given.

Figure 6.1: Network Representation of Transshipment Problem



Solving Transshipment Problem using Linear Programming

Let

X_{ij} be the number of units shipped from node i to node j,

X_{13} be the number of units shipped from Coimbatore to Chennai,

X_{24} be the number of units shipped from Pune to Mumbai, and so on

Table 6.2 shows the unit transportation cost from sources to destination.

Notes

Table 6.2: TP of the Shipment

	Warehouse	
Facility	**Chennai**	**Mumbai**
Coimbatore	4	7
Pune	6	3

Warehouses	**Retail outlets**			
	Delhi	**Hyderabad**	**Bangalore**	**Thiruvananthapuram**
Chennai	7	4	3	5
Mumbai	5	6	7	8

Objective: The objective is to minimize the total cost

Minimize $Z = 4X_{13} + 7X_{14} + 6X_{23} + 3X_{24} + 7X_{35} + 4X_{36} + 3X_{37} + 5X_{38} + 5X_{45}6X_{46} + 7X_{47} + 8X_{48}$

Constraints: The number of units shipped from Coimbatore must be less than or equal to 800. Because the supply from Coimbatore facility is 800 units. Therefore, the constraints equation is as follows:

$$X_{13} + X_{14} \leq 800 \quad \text{(i)}$$

Similarly, for Pune facility

$$X_{23} + X_{24} \leq 600 \quad \text{(ii)}$$

Now, considering the node 3,

Number of units shipped out from node 1 and 2 are,

$$X_{13} + X_{23}$$

Number of units shipped out from node 3 is,

$$X_{35} + X_{36} + X_{37} + X_{38}$$

The number of units shipped in must be equal to number of units shipped out, therefore

$$X_{13} + X_{23} = X_{35} + X_{36} + X_{37} + X_{38}$$

Bringing all the variables to one side, we get

$$-X_{13} - X_{23} + X_{35} + X_{36} + X_{37} + X_{38} = 0 \quad \text{(iii)}$$

Similarly for node 4

$$-X_{14} - X_{24} + X_{45}X_{46} + X_{47} + X_{48} = 0 \quad \text{(iv)}$$

Now considering the retail outlet nodes, the demand requirements of each outlet must be satisfied. Therefore for retail node 5, the constraint equation is

$$X_{35} + X_{45} = 350 \quad \text{(v)}$$

Similarly for nodes 6, 7, and 8, we get,

$$X_{36} + X_{46} = 200 \quad \text{(vi)}$$

$$X_{37} + X_{47} = 400 \quad \text{(vii)}$$

$$X_{38} + X_{48} = 450 \quad \text{(viii)}$$

Notes

Linear Programming formulation,

Minimize $Z = 4X_{13}+7X_{14}+6X_{22}+3X_{24}+7X_{35}+4X_{36}+3X_{37}+5X_{38}+5X_{45}+6X_{46}+7X_{47}+8X_{48}$

Subject to constraints,

$$\left.\begin{array}{l} X_{13}+X_{14} \le 800 \\ X_{23}+X_{23} \le 600 \end{array}\right\} \text{origin constraints}$$

$$-X_{13}-X_{23}+X_{35}+X_{36}+X_{37}+X_{38}=0$$

$$-X_{14}-X_{24}+X_{45}+X_{46}+X_{47}+X_{48}=0$$

$$\left.\begin{array}{l} X_{35}+X_{45}=350 \\ X_{36}+X_{46}=200 \\ X_{37}+X_{47}=400 \\ X_{38}+X_{48}=450 \end{array}\right\} \text{Destination constraints}$$

Caution **Don't confuse between transportation and transshipment problem**

A transportation problem can be converted into a transshipment problem by relaxing the restrictions on the receiving and sending the units on the origins and destinations respectively. A m-origin, n-destination, transportation problem, when expressed as transshipment problem, shall become an enlarged problem: with m + n origins and an equal number of destinations. With minor modifications, this problem can be solved using the transportation method.

Case Study

Transportation for Manufacturing Unit

A toy manufacturer wants to open a third warehouse that will supply three retail outlets. The new warehouse will supply **500 units** of backyard play sets per week. Two locations are being studied, N1 and N2. Refer to the table below for transportation costs per play set from each warehouse to each store locations.

	Store Location		
Warehouse	A	B	C
N1	$6	$8	$7
N2	$10	$6	$4

The existing system is shown on the following table.

	Store Location			
Warehouse	A	B	C	Capacity
1	$8	$3	$7	500
2	$5	$10	$9	400
Demand	400	600	400	

Question:

Which location would result in the lower transportation cost for the system?

Notes

Self Assessment

Fill in the blanks:

4. ……. is an extension of the transportation problem in which the commodity can be transported to a particular destination through one or more intermediate or transshipment nodes.
5. A very ……cost is applied to prohibited routes.
6. The objective of the transshipment problem is to determine how many units should be shipped over each node so that all the demand requirements are met with the minimum ……..

6.7 Summary

- An optimal solution is one where there is no other set of transportation routes that would reduce the total transportation cost, for which we have to evaluate each unoccupied cell (which represents unused routes) in terms of opportunity cost.
- Optimality can be tested by two ways namely:
  - Stepping Stone Method
  - Modified Distribution Method
- Stepping Stones is method for computing optimum solution of a transportation problem.
- The transshipment problem is an extension of the transportation problem in which the commodity can be transported to a particular destination through one or more intermediate or transshipment nodes.

6.8 Keywords

MODI Method: The modified distribution method, also known as MODI method or (u – v) method provides a minimum cost solution to the transportation problem.

Optimal Solution: A feasible solution is said to be optimal if it minimizes the total transportation cost.

Stepping Stone Method: In the stepping stone method, we have to draw as many closed paths as equal to the unoccupied cells for their evaluation. To the contrary, in MODI method, only closed path for the unoccupied cell with highest opportunity cost is drawn.

Transshipment: When it is possible to ship both into and out of the same node.

6.9 Review Questions

1. What are the conditions for forming a closed loop?
2. How are the maximization problems solved using transportation model?
3. How is optimality tested in solving transportation problems?
4. In what ways is a transshipment problem different from a transportation problem?

Notes

5. Solve the following transportation problems.

(a)

From	To			Available
	A	B	C	
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Requirement	4	2	2	

(b)

From	To			Available
	A	B	C	
I	6	8	4	14
II	4	9	8	12
III	1	2	6	5
Demand	6	10	15	

6. A potato chip manufacturer has three plants and four warehouses. Transportation cost for shipping from plants to warehouses, the plant availability and warehouses requirements are as follows:

Plants	Warehouses				Plant Availability (quintals)
	W_1	W_2	W_3	W_4	
F_1	7	4	3	5	235
F_2	6	8	7	4	280
F_3	5	6	9	10	110
Requirements (quintals)	125	160	110	230	625

Find optimum shipping schedule.

7. Solve

	Destination				Available
	1	2	3	4	
1	21	16	25	13	11
2	17	18	14	23	13
3	32	27	18	41	19
Requirement	6	10	12	15	43

8. Find the optimal solution.

	D_1	D_2	D_3	D_4	Available
O_1	23	27	16	18	30
O_2	12	17	20	51	40
O_3	22	28	12	32	53
Required	22	35	25	41	123

The cell entries are unit transportation cost.

Notes

9. Six plants with capacities of 100, 70, 80, 20, 55 and 95 respectively, supply goods to five stores requiring 50, 200, 30, 60 and 80 units respectively. The transportation costs matrix are given below. Find the optimal solution:

Plants	Stores				
	1	**2**	**3**	**4**	**5**
1	10	8	6	3	5
2	2	9	9	8	0
3	0	10	15	8	7
4	6	3	12	15	0
5	8	9	6	2	1
6	20	0	8	5	11

10. Find the minimum transportation cost.

Source	Destination					Available
	D_1	D_2	D_3	D_4	D_5	
S_1	4	7	3	8	2	4
S_2	1	4	7	3	8	7
S_3	7	2	4	7	7	9
S_4	4	7	2	4	7	2
Requirement	8	3	7	2	2	-

11. Solve the following transportation problem for maximum profit:

Warehouse	Per unit Profit (Rs.) Market				Availability of Warehouses
	A	B	C	D	
X	12	18	6	25	200
Y	8	7	10	18	500
Z	14	3	11	20	300
Demand in the Market	180	320	100	400	

Answers: Self Assessment

1. (a)
2. (c)
3. (b)
4. Transshipment
5. High
6. Transportation Cost

6.10 Further Readings

Notes

Books

J.K. Sharma, *Operations Research, Theory and Applications,* MacMillan India Ltd.

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Richard Bronson, Govindasami Naadimuthu, *Schaum's Outline of Theory and Problems of Operations Research,* McGraw-Hill Professional; 1997

Online links

http://myopm.net/courses/common/dt_module/tp_tutorial.pdf

www.nos.org

www.hkbu.edu.hk/

Notes

Unit 7: Assignment Problem – Balanced

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Objectives

After studying this unit, you will be able to:

- Understand the nature of assignment problem
- Analyze the mathematical formulation
- Learn the methods of solutions

Introduction

It is not uncommon to see Business Organisations confronting the conflicting need for optimal utilization of their limited resources among competing activities. In recent years, Linear Programming has received wider acclaim among the decision makers as a tool for achieving the business objectives. Out of various Quantitative Techniques developed over the past three decades, Linear Programming (LP) has found application in a wider screen. LP relates to the problems concerning distribution of scarce resources (satisfying some constraints which can be algebraically represented as Linear equations) so as to maximize profit or minimize cost. Under LP, decisions are arrived at under certainty conditions i.e., the information available on resources and relationship between variables are known. Hence, the course of action chosen will invariably lead to optimal or nearly optimal results.

The prominent problems which gained much importance under the house of LP are:

(1) Assignment problems.

(2) Transportation problems.

L.P is used in solving problems faced in assigning the 'equal number of jobs to equal number of workers so as to maximise profit or minimize cost'. Hence it is called one-to-one assignment. Say for instance, there are 'n' jobs to be performed and 'n' number of persons are available for doing these jobs and each person can do one job at a time though with varying degree of efficiency. Say let C_{ij} be the total cost : here C=cost, I = individual and j = job. So, a problem arises

Notes

as to which worker is to be assigned which job as to minimize the total job cost. The simple matrix would go like this:

Cost Matrix Job

Individual		1	2	J	n
	1	C_{11}	C_{12}	C_{1j}	C_{1n}
	2	C_{21}	C_{22}	C_{2j}	C_{2n}
	i	C_{i1}	C_{i2}	C_{ij}	C_{in}
	n	C_{n1}	C_{n2}	C_{nj}	C_{nn}

Although these types of problems could be solved by using transportation algorithm but a more efficient method called the assignment algorithm is used to solve such typical problems.

$C_{ij} \rightarrow$ indicates the cost of assigning i^{th} job to j^{th} individual

$X_{ij} \rightarrow$ (reference index) which indicates whether i^{th} job is assigned to j^{th} person or not.

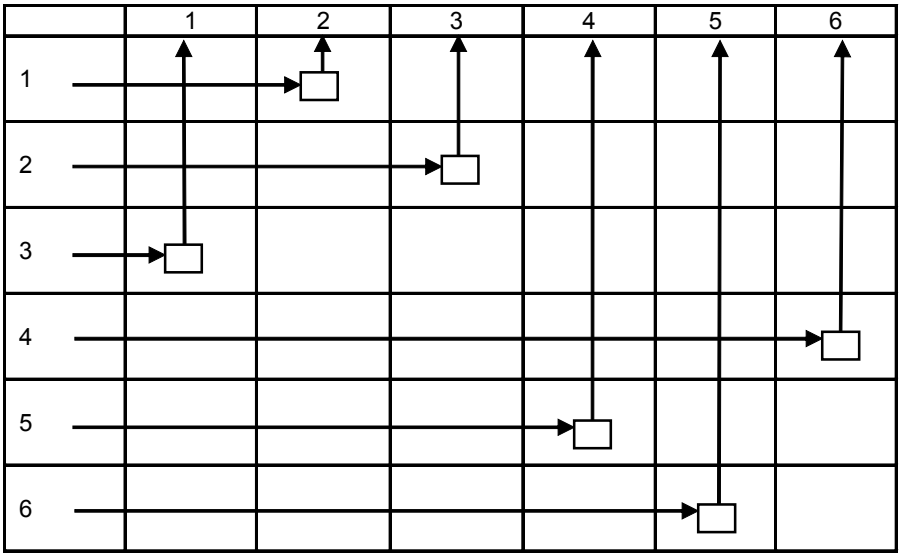
X_{ij} = 1 if i^{th} job are assigned to j^{th} person.

0 if i^{th} job are assigned to j^{th} person.

Consider the example:

There are 6 persons and 6 jobs to be allotted.

Let the assignment of jobs be as shown below:



The assignments are $1 \rightarrow 2$, $2 \rightarrow 3$, $3 \rightarrow 1$, $4 \rightarrow 6$, $5 \rightarrow 4$, $6 \rightarrow 5$.

Here, $x_{11} = 0$ since the first job is not allotted to first person.

$X_{12} = 1$ since the first job is allotted to second person.

Similarly, $x_{21} = x_{22} = x_{32} = x_{34} = 0$, and

$x_{31} = x_{46} = x_{54} = x_{65} = 1$

Sum of all jobs assigned to first person = $\sum_{i=1}^{6} X_{i1} = 1$

Notes

Since only one job can be allotted to a person.

$\sum_{j=1}^{6} X_{ij}$ = Sum of all persons with first job = 1, since a job can be assigned to only one person.

In general, sum of all jobs assigned to j^{th} person = 1

i.e., $\sum_{i=j}^{n} X_{ij} = 1$

And sum of all persons with ith job = 1

i.e., $\sum_{i=j}^{n} X_{ij} = 1$

And initial basic feasible solution can be found out by following:

1. Reduction Theorem
2. Hungarian Approach

Similarly, many real life problems can be solved such as assigning number of classes, for number of rooms, number of drivers to number of trucks or vice versa, number of teachers to number of classes, etc.

Reduction Theorem can be used for solving assignment problems with an objective of minimization of costs. For such maximization assignment problems, commonly used rules are:

1. Blind fold assignment/assignment by intuition.
2. Converting the maximization problem into minimization by considering the largest element in the whole matrix.
3. Converting the maximization problem into minimization by using negative signs for all the elements in the profit matrix.

Notes Assignment Problem is a variation of the transportation problem with two characteristics:

1. Cost matrix is a square matrix
2. The optimal solution for the problem would always be such that there would be only one assignment in a given row or column of the cost matrix

7.1 Application of Assignment Problem

Few applications of assignment problem are as follows:

1. Assignment of employees to machines.
2. Assignment of operators to jobs.
3. Effectiveness of teachers and subjects.
4. Allocation of machines for optimum utilization of space.

Notes

5. Allocation of salesmen to different sales areas.
6. Allocation of clerks to various counters.

In all the cases, the objective is to minimize the total time and cost or otherwise maximize the sales and returns.

Self Assessment

Multiple Choice Questions:

1. An assignment problem will have the following solution
   (a) optimal (b) unique
   (c) multiple (d) all of the above
2. Maximization assignment problem is transformed into minimization problem by
   (a) Adding each entry in a column from the maximum value in that column
   (b) Subtracting each entry in a column from maximum value in that column
   (c) Subtracting each entry in the table from the maximum value in that table
   (d) Any one of the above
3. The main objective of an assignment problem is to
   (a) Minimize the total cost (b) Maximize the sales and returns.
   (c) Both (a) and (b) (d) None

7.2 Types of Assignment Problem

The assignment problems are of two types. It can be either

(i) Balanced or

(ii) Unbalanced.

If the number of rows is equal to the number of columns or if the given problem is a square matrix, the problem is termed as a *balanced assignment problem*. If the given problem is not a square matrix, the problem is termed as an *unbalanced assignment problem*.

If the problem is an unbalanced one, add dummy rows / dummy columns as required so that the matrix becomes a square matrix or a balanced one. The cost or time values for the dummy cells are assumed as zero.

Self Assessment

Fill in the blanks:

4. An assignment problem is said to be unbalanced when .....
5. When the number of rows is equal to the number of columns then the problem is said to be ..... assignment problem.
6. For solving an assignment problem the matrix should be a ..... matrix.

Notes

7.3 Mathematical Model of Assignment Problem

Minimize the total cost which is given by,

$$Z = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}X_{ij}$$

Where, i = 1, 2, 3,……n

j = 1, 2, 3,……n

Subject to restriction

$X_{ij} = 1$ (One job is done by one worker)

$= 0$ (No job is assigned)

$\Sigma X_{ij} = 1$ (only one job be assigned to one person) Where, j = 1,2, 3, ……n

$\Sigma X_{ij} = 1$ (only one person can do one job at a time)

Where, i=1, 2, 3,…….n

$C_{ij} \rightarrow$ indicates the cost of assigning i^{th} job to j^{th} individual or vice versa, Vi_{ij} = 1 to n.

$X_{ij} \rightarrow$ indicates whether i^{th} job is assigned to j^{th} person or not.

X_{ij} = 1 if i^{th} job is assigned to j^{th} person '0' otherwise.

Theorem Statement

It states that in an assignment problem if we add or subtract a constant to every element of any row or column of the cost matrix (C_{ij}), then an assignment that minimizes the total cost on one matrix will also minimize the total cost on the other matrix.

Proof:

Let $X_{ij} = X_{ij}$ X_{ij} = elements of first cost matrix.

X_{ij} = elements of second cost matrix.

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}X_{ij} : X_{ij} \geq 0$$

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(C_{ij} \pm X_{ij} \pm V_j\right) X_{ij}$$

X_{ij} = i = 1.2.3…….n

j = 1.2.3…….n

$U_i \rightarrow i^{th}$ row constant taken for reduction.

$V_j \rightarrow j^{th}$ column constant taken for reduction.

Where, U_i and V_j are considered to be constant.

$$Z_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}X_{ij} - \sum U_i \sum_{i-1}^{n} X_{ij} - \sum_{j=1}^{n} V_j \sum_{j=1}^{n} X_{ij}$$

Notes

$$Z_1 = Z - \sum_{i-1}^{n} U_i - \sum_{j=1}^{n} V_j$$

$$\sum_{j=1}^{n} X_{ij} = \sum_{i=1}^{n} x_{ij} = 1$$

Since, terms that are subtracted from 'Z' to give 'Z_1' are independent of x_{ij}, it follows that 'Z' is minimized whenever 'Z_1' is minimized and it can be proved conversely.

Notes In an assignment problem if a constant is added to or subtracted from every element of any row or column of the cost matrix, an assignment that minimizes the total cost in one matrix also minimizes the total cost in the other matrix.

Example: **Reduction Theorem**

Find the minimum cost for the following problem:

	Persons			
	I	**II**	**III**	**IV**
A	10	12	19	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

Tasks

(i) Row-wise reduction

	I	II	III	IV
A	0	2	9	1
B	0	5	2	3
C	1	3	2	0
D	0	7	3	1

(ii) Column-wise reduction

	I	II	III	IV
A	0	$\boxed{0}$	7	1
B	0	3	$\boxed{0}$	3
C	1	1	0	0
D	$\boxed{0}$	5	1	1

Hence, the assignment is

A	→	II	12
B	→	III	07
C	→	IV	11
D	→	I	08
			₹ 38

Here minimized cost $= \Sigma\Sigma\, C_{ij}X_{ij}$ = ₹ 38

Example: Solve the following assignment problem and find the minimum cost.

Notes

Solve the following assignment problem and find the minimum cost.

Jobs		I	II	III	IV
Workers	A	10	12	19	11
	B	5	10	7	8
	C	12	14	13	11
	D	8	15	11	9

Solution:

Using Reduction Rules

Step 1: Row-wise Reduction of the matrix.

Jobs		I	II	III	IV
Workers	A	0	2	9	1
	B	0	5	2	3
	C	1	3	2	0
	D	0	7	3	1

Step 2: Column-wise Reduction of the matrix.

Jobs		I	II	III	IV
Workers	A	0	0	7	1
	B	0	3	0	3
	C	1	1	0	0
	D	0	5	1	1

Step 3: Assignment of jobs to workers.

Jobs		I	II	III	IV
Workers	A	0	$\boxed{0}$	7	1
	B	0	3	$\boxed{0}$	3
	C	1	1	0	$\boxed{0}$
	D	$\boxed{0}$	5	1	1

Step 4: Calculation of the Minimum total job cost associated with the assignment.

Workers	Jobs	Total Cost (Rs.)
A	II	12
B	III	7
C	IV	11
D	I	8
Therefore Total job cost		38

Inference

The minimum total cost required to complete the assignment is ₹ 38.

Notes

Example: A departmental head has 4 subordinates and 4 tasks are to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effective matrix. How the tasks should be allocated to each person so as to minimize the total man hours?

Subordinates		I	II	III	IV
Tasks	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

Solution:

Using Reduction Theorem Rules

Step 1: Row-wise reduction of the matrix.

Subordinates		I	II	III	IV
Tasks	A	0	18	9	3
	B	9	24	0	22
	C	23	4	3	0
	D	9	16	14	0

Step 2: Column-wise reduction of the matrix.

Subordinates		I	II	III	IV
Tasks	A	0	14	9	3
	B	9	20	0	22
	C	23	0	3	0
	D	9	12	14	0

Step 3: Assignment of tasks to subordinates.

Subordinates		I	II	III	IV
Tasks	A	$\boxed{0}$	14	9	3
	B	9	20	$\boxed{0}$	22
	C	23	$\boxed{0}$	3	0
	D	9	12	14	$\boxed{0}$

Step 4: Calculation of the minimum total man hours associated with the assignment.

Subordinates	Tasks	Total Man Hours
I	A	8
II	C	19
III	B	4
IV	D	10
		41

Notes

Inference

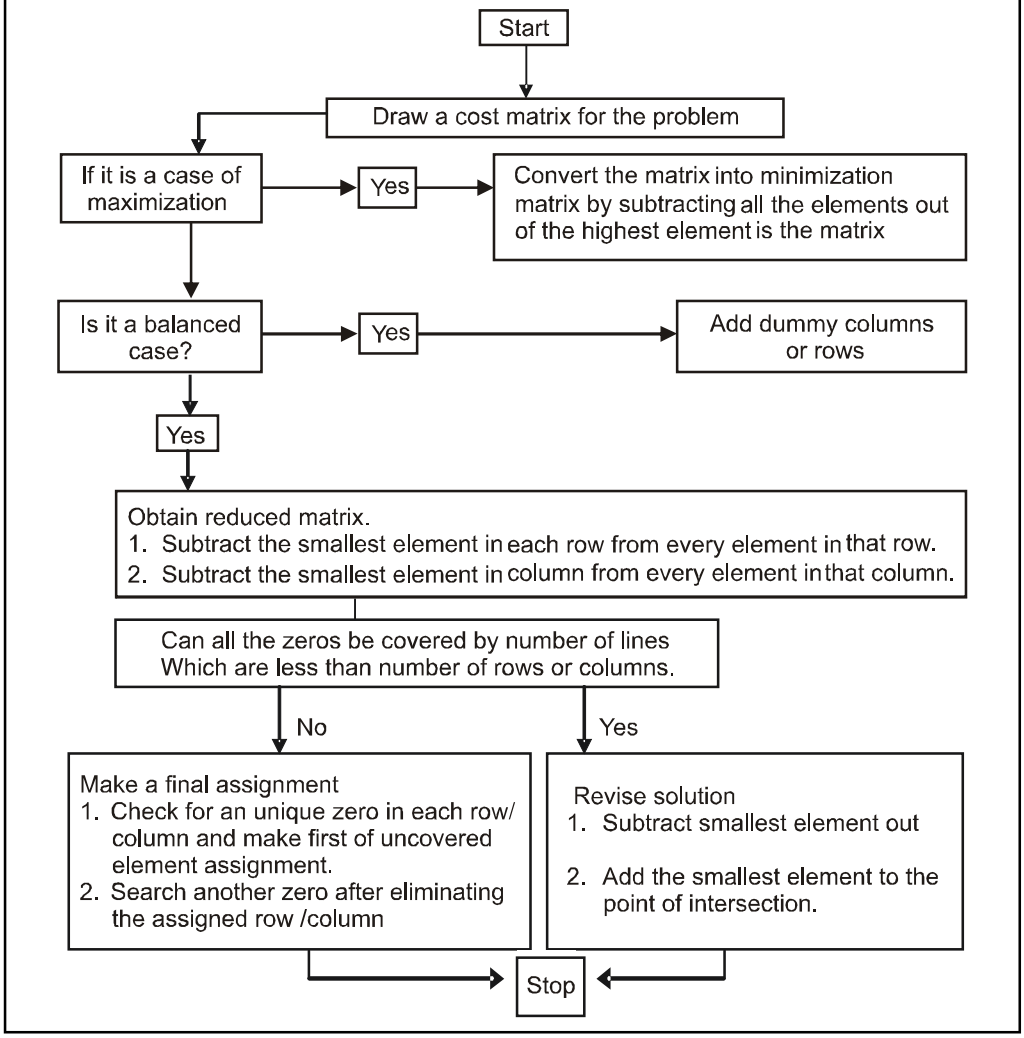
Therefore, the minimum man hours required to complete the assignment are 41.

Task Show that the optimal solution of an assignment problem is unchanged if we add or subtract the same constant to the entries of any row or column of the cost matrix.

7.4 Hungarian Approach

The systematic procedure to be followed while solving assignment problems under Hungarian Approach is picturized in the chart shown below:

Figure 5.1: Diagrammatic Representation of Hungarian Approach



The credit of developing an algorithm goes to Hungarian mathematician D. Konig and later on, it became popular as Hungarian method. This technique is used to solve more typical problems, viz., assignment of chines to jobs, assignment of salesmen to sales territories, assignment of workers to various tasks, vehicles to routes, contracts to bidders, etc.

Notes

Hungarian method of solving assignment problems is based on two important properties:

1. In given cost matrix, if a constant quantity is added or subtracted from every element of any row or column, an assignment that minimizes the total cost in one matrix also minimizes the total cost in the other.
2. For an assignment problem having all non-negative cost, a solution having zero total cost is an optimum solution.

Example: **Hungarian Approach**

Let U_i denote the ith row constant which is added to i^{th} row elements and V_j denote the jth column constant which is added to elements of jth column.

Tasks

	I	II	III	IV	U_i
A	10	12	19	11	$U_1 = 10$
B	5	10	7	8	$U_2 = 5$
C	12	14	13	11	$U_3 = 8$
D	8	15	11	9	$U_4 = 2$
V_j	$V_1 = 5$	$V_2 = 8$	$V_3 = 7$	$V_4 = 9$	

$$\text{Total } U_i = \Sigma U_i = 25$$

$$\text{Total } V_j = \Sigma V_j = 29$$

First of all, we will add U_{is} to all the rows.

	I	II	III	IV
A	20	22	29	21
B	10	15	12	13
C	20	22	21	19
D	10	17	13	11
Vj	V1 = 5	V2 = 8	V3 = 7	V4 = 9

Now, we shall add V_{js} to all the columns.

	I	II	III	IV
A	25	20	36	30
B	15	23	19	22
C	25	30	28	28
D	15	25	20	20

Now, we will find the minimum cost of assignment for this problem.

	I	II	III	IV
A	0	5	11	5
B	0	8	4	7
C	0	5	3	3
D	0	10	5	5

Notes

Column reduction.

	I	II	III	IV
A	0	0	7	2
B	0	3	1	4
C	0	0	0	0
D	0	5	2	2

L_2

L_1

Here, we use Hungarian method to find the optimum solution.

	I	II	III	IV
A	1	$\boxed{0}$	8	2
B	0	2	$\boxed{0}$	3
C	1	0	0	$\boxed{0}$
D	$\boxed{0}$	4	1	1

Here, assignment is

A	→	II	30
B	→	III	19
C	→	IV	28
D	→	I	15

The minimum cost is ₹ 92

Hence, $Z_1 = 92$

We got the cost matrix for Z_1 by adding U_1 to different rows and V_j to different columns of cost matrix for Z.

Hence, $Z_1 = Z + \Sigma U_i + \Sigma V_j$

i.e., $92 = 38 + 25 + 29$

Observe that the assignment for both the matrix is same. Hence, we can say that Z is minimized whenever Z_1 is minimized. However, the solution (minimised cost for Z = and Z_1) will be different due to cost elements (C_{ij}).

Example: In a textile sales emporium, 4 sales girls Arpitha (A1), Archana (A2), Aradhana (A3) and Aakansha (A4) are available to 4 sales counters M, N, O and P. Each sales girl can handle any counter. The service of each sales counter [in hours] when carried out by each sales girl is given below:

Sales girls		**A_1**	**A_2**	**A_3**	**A_4**
Sales counters	M	41	72	39	52
	N	22	29	49	65
	O	27	39	60	51
	P	45	50	48	52

How to allocate the appropriate sales counters to sales girls so as to minimize the service time?

Solution:

Notes

Using Reduction Rules

Step 1: Row-wise reduction of the matrix.

Sales girls		A1	A2	A3	A4
Sales counters	M	2	33	0	13
	N	0	7	27	43
	O	0	12	33	24
	P	0	5	3	7

Step 2: Column-wise reduction of the matrix.

Sales girls		A1	A2	A3	A4
Sales counters	M	$\boxed{2}$	28	0	6
	N	0	2	27	36
	O	0	7	33	17
	P	$\boxed{0}$	0	3	0
		L_1			L_2

Step 3: Hungarian Approach used.

Sales girls		A1	A2	A3	A4
Sales counters	M	4	28	$\boxed{0}$	6
	N	0	$\boxed{0}$	25	34
	O	$\boxed{0}$	5	31	15
	P	2	0	3	0

Step 4: Trial assignment.

Sales girls		A1	A2	A3	A4
Sales counters	M	4	28	$\boxed{0}$	6
	N	0	$\boxed{0}$	25	34
	O	$\boxed{0}$	5	31	15
	P	2	0	3	$\boxed{0}$

Step 5: Calculation of total time on optimal assignment.

Sales Counters	Sales girls	Total sales Time
M	A3	39
N	A2	29
O	A1	27
P	A4	52
		147

Notes

Step 6: Requirement 2: By intuition.

Sales girls		A1	A2	A3	A4
Sales counters	M	41	72	39	52
	N	22	29	49	65
	O	27	39	60	51
	P	45	50	48	52

Step 7: Calculation of total time on optimal assignment.

Sales Counters	Sales girls	Total sales Time
M	A3	39
N	A1	22
O	A2	39
P	A4	52
		147

Inference

The test of optimality can be confirmed when an assignment is carried out by intuition. As it is apparent from the above revealed results, that the assignment by intuition would come out with 152 hours. As against hours when Hungarian approach is used.

Example: A project consists of 4 major jobs for which 4 contractors have submitted their tenders. The tender amount quoted in lakhs of rupees are given in the matrix below:

Jobs		A	B	C	D
Contractors	1	10	24	30	15
	2	16	22	28	12
	3	12	20	32	10
	4	9	26	34	16

Find the assignment which minimizes the total project cost [each contractor has to be assigned at least one job].

Solution:

Using reduction rules

Step 1: Row-wise reduction of the matrix.

Jobs		A	B	C	D
Contractors	1	0	14	20	5
	2	4	10	16	0
	3	2	10	22	0
	4	0	17	25	7

Notes

Step 2: Calculation-wise reduction of the matrix.

Jobs		A	B	C	D
Contractors	1	0	4	4	5
	2	4	0	0	0
	3	2	0	6	0
	4	0	7	9	7

Step 3: Applying Hungarian approach.

Jobs		A	B	C	D
Contractors	1	0	4	4	5
	2	4	0	0	0
	3	2	0	6	0
	4	0	7	9	7

Step 4: Trial assignment.

Jobs		A	B	C	D
Contractors	1	0	0	$\boxed{0}$	1
	2	8	0	0	$\boxed{0}$
	3	6	$\boxed{0}$	6	0
	4	$\boxed{0}$	3	5	2

Step 5: Calculation of total minimum cost.

Job	Contractors	Total Cost (Rs.)
A	4	09
B	3	20
C	1	30
D	2	12
		71

Inference

Therefore, the total minimum cost required to complete the assignment, i.e., 4 jobs by 4 contractors is ₹ 71 (i.e., ₹ 71,00,000).

Did u know? An assignment problem can be solved by the following four methods:

1. Enumeration Method
2. Simplex Method
3. Transportation Method
4. Hungarian Method

Notes

Caution Assignment Problem may not necessarily have an optimal solution. It may also have multiple solutions to a given problem.

7.5 Summary

- An assignment problem seeks to minimize the total cost assignment of n workers to n jobs.
- It assumes all workers are assigned and each job is performed.
- An assignment problem is a special case of a transportation problem in which all supplies and all demands are equal to 1; hence assignment problems may be solved as linear programs.
- Applications of assignment problems are varied in the real world.
- It can be useful for the classic task of assigning employees to tasks or machines to production jobs, but its uses are more widespread.

7.6 Keywords

Allocate: To distribute according to a plan; allot

Assignment Problem: Assignment problem is used to assign *n* number of resources to *n* number of activities so as to minimize the total cost or to maximize the total profit of allocation in such a way that the measure of effectiveness is optimized.

7.7 Review Questions

1. Describe the assignment problem giving a suitable example. Give two areas of its application.
2. Explain the difference between a transportation problem and an assignment problem.
3. Give the mathematical formulation of the assignment problem.
4. Show that the optimal solution of an assignment problem is unchanged if we add or subtract the same constant to the entries of any row or column of the cost matrix.
5. Describe the method of drawing minimum number of lines in the context of assignment problem. Name the method.

Problems

6. A firm plans to begin production of three new products. They own three plants and wish to assign one new plant. The unit cost of producing i at plant j is Cij as given by the following matrix. Find the assignment that minimizes the total unit cost.

$$\begin{array}{cc} & \text{Plants} \\ \text{Product} & \begin{pmatrix} 10 & 8 & 12 \\ 18 & 6 & 14 \\ 6 & 4 & 2 \end{pmatrix} \end{array}$$

Notes

7. A company has 4 machines on which do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table:

Job	Machine			
	A	**B**	**C**	**D**
1	18	24	28	32
2	8	13	17	19
3	10	15	19	22

Determine the optimum assignment.

8. Solve.

(a)

Jobs	Men			
	A	**B**	**C**	**D**
I	1	4	6	3
II	9	7	10	9
III	4	5	11	7
IV	8	7	8	5

(b)

Jobs	Men			
	A	**B**	**C**	**D**
I	10	25	15	20
II	15	30	5	15
III	35	20	12	24
IV	17	25	24	20

9. A machine tool company decides to make 4 sub-assemblies through four contractors. Each contractor is to receive only one sub-assembly. The cost of each sub-assembly is determined by the bids submitted by each contractor and is shown in table below (in '000 ₹). Assign different assemblies to contractors so as minimize the total cost:

Sub-assembly	Contractor			
	A	**B**	**C**	**D**
I	15	13	14	17
II	11	12	15	13
III	18	12	10	11
IV	15	17	14	16

Notes

10. Solve and find the maximum total expected sale.

Salesman	Area			
	I	II	III	IV
A	42	35	28	21
B	30	25	20	15
C	30	25	20	15
D	24	20	16	12

11. The following is the cost matrix of assigning 4 clerks to keypunching jobs. Find the optimal assignment if clerk 1 cannot be assigned to job 1.

Clerk	Job			
	1	2	3	4
1	-	5	2	0
2	4	7	5	6
3	5	8	4	3
4	3	6	6	2

12. A student has to select one and only one elective in each semester and the some elective should not be selected in different semesters. Due to various reasons, the expected grades in each subject, if selected in different semester, vary and they are given below.

Semester	Advanced OR	Advanced Statistics	Graph Theory	Discrete Mathematics
I	F	E	D	C
II	E	E	C	C
III	C	D	C	A
IV	B	A	H	H

The grade points are; H = 10, A = 9, B = 8, C = 7, D = 6, E = 5 and F = 4. How will the student select the electives in order to maximize the total expected point's and what will be his maximum expected total points?

13. Kapil Corporation has four plants each of which can manufacture any of the products. Production costs differ from one plant to another plant as does revenue. Given the revenue and cost data below, obtain which product each plant should produce to maximize profit.

Plant			Sales Revenue (₹ '000) Product			Production Cost (₹ '000) Product		
	1	2	3	4	1	2	3	4
A	50	68	49	62	49	60	45	61
B	60	70	51	74	55	63	45	69
C	55	67	53	60	52	62	49	58
D	68	65	64	69	55	64	48	66

Notes

14. Consider the problem of assigning five jobs to 4 persons.

Person	Job				
	1	2	3	4	5
A	8	4	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

Answers: Self Assessment

1. (d)
2. (c)
3. (c)
4. number of rows is not equal to the number of columns
5. Balanced
6. Square

7.8 Further Readings

Books

J.K. Sharma, *Operations Research, Theory and Applications,* MacMillan India Ltd.

Kanti Swarup, P.K Gupta & Manmohan, *Operations Research*, Sultan Chand Publications, New Delhi

Michael W. Carter, Camille C. Price, *Operations Research: A Practical Introduction,* CRC Press, 2001

Paul A. Jensen, Jonathan F. Bard, *Operations Research Models and Methods,* John Wiley and Sons, 2003

Richard Bronson, Govindasami Naadimuthu, *Schaum's Outline of Theory and Problems of Operations Research,* McGraw-Hill Professional; 1997

Online links

www.utdallas.edu/

http://www.usna.edu/

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Notes

Unit 8: Assignment Problem – Unbalanced

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Objectives

After studying this unit, you will be able to:

- Know about variations of the assignment problem
- Learn how to solve an unbalanced assignment problem, profit maximization problem etc.
- Understand how to solve a travelling salesman problem

Introduction

The basic objective of an assignment problem is to assign n number of resources to n number of activities so as to minimize the total cost or to maximize the total profit of allocation in such a way that the measure of effectiveness is optimized.

The problem of assignment arises because available resources such as men, machines, etc., have varying degree of efficiency for performing different activities such as job. Therefore cost, profit or time for performing the different activities is different. Hence the problem is, how should the assignments be made so as to optimize (maximize or minimize) the given objective.

The assignment model can be applied in many decision-making processes like determining optimum processing time in machine operators and jobs, effectiveness of teachers and subjects, designing of good plant layout, etc. This technique is found suitable for routing traveling salesmen to minimize the total traveling cost, or to maximize the sales.

In this unit you will learn how to solve special types of assignment problems such as unbalanced problems, travelling salesman problem etc.

Notes

8.1 Variations of the Assignment Problem

There can be many types of variations in a assignment problem. In this unit we will discuss in detail about the various types of assignment problem you may encounter in the real life scenario.

In this unit, we will discuss on following four variations of the assignment problem:

1. Multiple Optimal Solutions
2. Maximization case in assignment problems
3. Unbalanced Assignment problem
4. Travelling Salesman Problem

8.2 Multiple Optimal Solutions

While making an assignment in the reduced assignment matrix, it is possible to have two or more ways to strike off a certain number or zeroes. Such a situation indicates multiple optimal solutions with the same optimal value of objective function. In such cases the more suitable solution may be considered by the decision maker.

Notes If the assignment problem has only one solution then the solution is said to be ***Unique solution.***

8.3 Maximization Assignment Problems

In maximization problem, the objective is to maximize profit, revenue, etc. Such problems can be solved by converting the given maximization problem into a minimization problem.

1. Change the signs of all values given in the table or another method is,
2. Select the highest element in the entire assignment table and subtract all the elements of the table from the highest element.

Alpha Corporation has 4 plants, each of which can manufacture any one of the 4 products. Production cost differs from one plant to another plant, so also the sales revenue. Given the revenue and the cost data below, obtain which product each plant should produce to maximize the profit.

Sales revenue (₹ in 000s)

Product		1	2	3	4
Plant	A	50	68	49	62
	B	60	70	51	74
	C	55	67	53	70
	D	58	65	54	69

Notes

Production Cost (₹ in' 000s)

Product		1	2	3	4
Plant	A	49	60	45	61
	B	55	63	45	69
	C	52	62	49	68
	D	55	64	48	66

Solution:

Step 1: Determination of profit matrix.

Product		1	2	3	4
Plant	A	1	8	4	1
	B	5	7	6	5
	C	3	5	4	2
	D	3	1	6	3

Step 2: Conversion of profit matrix into cost matrix.

Product		1	2	3	4
Plant	A	7	0	4	7
	B	3	1	2	3
	C	5	3	4	6
	D	5	7	2	5

Using Reduction Rules

Step 3: Row-wise reduction of the matrix.

Product		1	2	3	4
Plant	A	7	0	4	7
	B	2	0	1	2
	C	2	0	1	3
	D	3	5	0	3

Step 4: Column-wise reduction of the matrix.

Product		1	2	3	4
Plant	A	5	0	4	5
	B	0	0	1	0
	C	0	0	1	1
	D	1	5	0	1

Step 5: Trial Assignment.

Product		1	2	3	4
Plant	A	5	$\boxed{0}$	4	5
	B	0	0	1	$\boxed{0}$
	C	$\boxed{0}$	0	1	1
	D	1	5	$\boxed{0}$	1

Notes

Step 6: Determination of profit associated with the assignment.

Plant	Product	Total Profit (Rs.)
A	2	8,000
B	4	5,000
C	1	3,000
D	3	6,000
	Total Profit	22,000

When negative signs are used to make the optimal assignment.

Step 1: Profit matrix with (–)ve signs.

Product		1	2	3	4
Plant	A	–1	–8	–4	–1
	B	–5	–7	–6	–5
	C	–3	–5	–4	–2
	D	–3	–1	–6	–3

Step 2: Row-wise reduction of the matrix.

Product		1	2	3	4
Plant	A	7	0	4	7
	B	2	0	1	2
	C	2	0	1	3
	D	3	5	0	3

Step 3: Column-wise reduction of the matrix.

Product		1	2	3	4
Plant	A	5	0	4	5
	B	0	0	1	0
	C	0	0	1	1
	D	1	5	0	1

Step 4: Trial assignment.

Product		1	2	3	4
Plant	A	5	$\boxed{0}$	4	5
	B	0	0	1	$\boxed{0}$
	C	$\boxed{0}$	0	1	1
	D	1	5	$\boxed{0}$	1

Step 5: Determine of profit associated with the assignment.

Plant	Product	Total Profit (₹)
A	2	8,000
B	4	5,000
C	1	3,000
D	3	6,000
	Total Profit	22,000

Notes

Inference

Hence to get a maximum profit of ₹ 22,000, Alpha Corporation should produce 2 products in plant A, B should manufacture product 4, C plant should manufacture product 1 and D plant should manufacture product 3.

Example: A company has 4 territories and 4 salesmen available for assignment. The territories are not equally rich in their sales potential; it is estimated that a typical salesman operating in each territory would bring in the following annual sales:

Territory	I	II	III	IV
Annual Sales (₹)	60,000	50,000	40,000	30,000

Four salesmen are also considered to differ in their ability. It is estimated that working under the same conditions, their yearly sales would be proportionately as follows:

Salesmen	A	B	C	D
Proportion	7	5	5	4

If the criterion is the maximum expected total sales cum the intuitive answer is to assign the best salesman to the richest territory, the next best salesman to the second richest and so on. Verify this answer by assignment model.

Solution:

Step 1: Matrix showing the proportion of sales revenue.

Territory (in'000)		I (6)	II (5)	III (4)	IV (3)
Sales	A (7)	42	35	28	21
	B (5)	30	25	20	15
	C (5)	30	25	20	15
	D (4)	24	20	16	12

Step 2: Calculation of sales revenue associated with the assignment based on intuition.

Salesman	Territory	Total Profit (₹)
A	I	42,000
B	II/III	25,000/20,000
C	III/II	20,000/25,000
D	IV	12,000
		99,000

Step 3: Verification of the results intuitively found with the use of assignment model.

Conversion of sales revenue matrix into cost matrix.

Territory		I	II	III	IV
Salesman	A	0	7	14	21
	B	12	17	22	27
	C	12	17	22	27
	D	18	22	26	30

Notes

Step 4: Using reduction rules.

Row-wise reduction of the matrix.

Territory		I	II	III	IV
Salesman	A	0	7	14	21
	B	0	5	10	15
	C	0	5	10	15
	D	0	4	8	12

Step 5: Column-wise reduction of the matrix.

Territory		I	II	III	IV
Salesman	A	0	3	6	9
	B	0	1	2	3
	C	0	1	2	3
	D	0	0	0	0

Note: The above matrix which would have been used for making a trial assignment is not an effective matrix because all the zeros either fall on a row or a column and further the matrix cannot be reduced because each row has a zero element and each column has a zero element respectively. Hence, Hungarian approach is to be used to overcome this limitation.

Hungarian Approach

Step 1:

Territory		I	II	III	IV
Salesman	A	0	3	6	9
	B	0	1	2	3
	C	0	1	2	3
	D	$\boxed{0}$	0	0	0
		L_1			L_2

Step 2:

Territory		I	II	III	IV
Salesman	A	0	2	5	8
	B	0	0	1	2
	C	0	0	1	2
	D	1	0	0	0

Step 3: Further reduction of the matrix.

Territory		I	II	III	IV
Salesman	A	0	2	5	8
	B	0	0	1	2
	C	0	0	1	2
	D	$\boxed{0}$	$\boxed{0}$	0	0
		L_1	L_2		L_3

Notes

Step 4: Trial assignment.

Territory		I	II	III	IV
Salesman	A	0	2	4	6
	B	0	0	0	1
	C	0	0	0	1
	D	2	0	0	0

Step 5: Calculation of total sales revenue.

Salesman	Territory	Total Sales Revenue (Rs.)
A	I	42,000
B	II/III	25,000/20,000
C	III/II	20,000/25,000
D	IV	12,000
		99,000

Inference

Thus, by following the above assignment schedule for allocating the territories to the 4 salesmen, the sales revenue can be maximized to ₹ 99,000.

Task A marketing manager has 5 salesman and 5 sales districts. Considering the capabilities of the salesman and the nature of the districts, the marketing manager estimates that sales per month (in, 00 ₹) for each salesman in each district would be as follows:

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment that will result in maximum sale.

Self Assessment

Multiple Choice Questions:

1. An assignment problem will have the following solution:
   (a) optimal (b) unique
   (c) multiple (d) all of the above
2. Maximisation assignment problem is transformed into minimization problem by
   (a) adding each entry in a column from the maximum value in that column
   (b) subtracting each entry in a column from maximum value in that column
   (c) subtracting each entry in the table from the maximum value in that table
   (d) any one of the above

Notes

3. When an assignment problem has more than one solution, then it is

   (a) Multiple Optimal solution (b) The problem is unbalanced

   (c) Maximization problem (d) Balanced problem

8.4 Unbalanced Assignment Problem

If the given matrix is not a square matrix, the assignment problem is called an **unbalanced problem.** In such type of problems, add dummy row(s) or column(s) with the cost elements as zero to convert the matrix as a square matrix. Then the assignment problem is solved by the Hungarian method.

Example: A company has 4 machines to do 3 jobs. Each job can be assigned to 1 and only 1 machine. The cost of each job on a machine is given to the following table:

Machines		W	X	Y	Z
Jobs	A	18	24	28	32
	B	8	13	17	18
	C	10	15	19	22

What are the job assignments which will minimize the cost?

Solution:

Step 1: Conversion of the above unbalanced matrix into a balanced matrix.

Machines		W	X	Y	Z
Jobs	A	18	24	28	32
	B	8	13	17	18
	C	10	15	19	22
	D	0	0	0	0

Using reduction rules

Step 2: Row-wise reduction of the matrix.

Machines		W	X	Y	Z
Jobs	A	0	6	10	14
	B	0	5	9	10
	C	0	5	9	12
	D	0	0	0	0

Step 3: Column-wise reduction of the matrix.

Note: It is apparent from the above matrix that it is not possible to reduce the matrix column-wise. Hence, Hungarian approach is used.

Notes

Step 4: Hungarian approach is used.

Machines		W	X	Y	Z
Jobs	A	0	6	10	14
	B	0	5	9	10
	C	0	5	9	12
	D	0	0	0	0

L_1 L_2

Step 5:

Machines		W	X	Y	Z
Jobs	A	0	1	5	9
	B	0	0	4	5
	C	0	0	4	7
	D	5	0	0	0

Step 6: Trial assignment.

Machines		W	X	Y	Z
Jobs	A	0	1	1	5
	B	0	0	0	1
	C	0	0	0	3
	D	9	4	0	0

Step 7: Calculation of total minimum cost.

Jobs	Machines	Total cost (₹)
A	W	18
B	X/Y	13/17
C	Y/X	19/15
D	Z	0
	Minimum Time required	50

Inference

Therefore, the minimum cost associated with the assignment of 3 jobs to 4 machines is ₹ 50.

Example: Find the assignment of machines to jobs that will result in a maximum profit and which is the job that should be declined from the following. The owner of a small machine shop has 4 machines available to assign 2 jobs for the day, 5 jobs are offered with the expected profit (₹) per machinist on each job being as follows.

Job		A	B	C	D	E
Machinist	1	6.20	7.80	5	10.10	8.20
	2	7.10	8.40	6.10	7.30	5.90
	3	8.70	9.20	11.10	7.10	8.10
	4	4.80	6.40	8.70	7.70	8.00

Notes

Solution:

Step 1: Conversion of the unbalanced matrix into a balanced matrix.

Job		A	B	C	D	E
Machinist	1	6.20	7.80	5	10.10	8.20
	2	7.10	8.40	6.10	7.30	5.90
	3	8.70	9.20	11.10	7.10	8.10
	4	4.80	6.40	8.70	7.70	8.00
	5	0	0	0	0	0

Step 2: Conversion of maximisation matrix into a minimization case.

Job		A	B	C	D	E
Machinist	1	–6.20	–7.80	–5	–10.10	–8.20
	2	–7.10	–8.40	–6.10	–7.30	–5.90
	3	–8.70	–9.20	–11.10	–7.10	–8.10
	4	–4.80	–6.40	–8.70	–7.70	–8.00
	5	0	0	0	0	0

Step 3: Conversion of above matrix into an effective matrix.

Job		A	B	C	D	E
Machinist	1	3.90	2.30	5.10	0	1.90
	2	1.30	0	2.30	1.10	2.50
	3	2.40	1.90	0	4.0	3.00
	4	3.90	2.30	0	1.00	0.70
	5	0	0	0	0	0

It is apparent that in the above matrix the reduction rules cannot be applied hence direct Hungarian approach is followed.

Step 4: Hungarian approach is applied.

Job		A	B	C	D	E
Machinist	1	3.90	2.30	5.10	0	1.90
	2	1.30	0	2.30	1.10	2.50
	3	2.40	1.90	0	4.0	3.00
	4	3.90	2.30	0	1.00	0.70
	5	0	0	0	0	0
			L_1	L_2	L_3	L_4

Notes

Step 5: Trial assignment.

Job		A	B	C	D	E
Machinist	1	3.20	2.30	5.10	$\boxed{0}$	1.20
	2	0.60	$\boxed{0}$	2.30	1.10	1.80
	3	1.70	1.90	$\boxed{0}$	4.00	2.30
	4	3.20	2.30	0	1.00	$\boxed{0}$
	5	$\boxed{0}$	0.70	0.70	0.70	0.70

Step 6: Calculation of maximum profit for the assignment.

From	To	Total Sales (₹)
A	5	0
B	2	8.40
C	3	11.10
D	1	10.10
E	4	8.00
	Total profit	37.60

Inference

The maximum total profit for the assignment is ₹ 37.60. And the job to be declined is A as the machinist available 5th is supposed to be fictitious.

Self Assessment

Fill in the blanks:

4. If the given matrix is not a ……matrix, the assignment problem is called an unbalanced problem
5. A dummy row(s) or column(s) with the cost elements as …….. is added to convert the matrix of an unbalanced assignment problem as a square matrix.
6. A given assignment problem can be classified into balanced or unbalanced on the basis of ……

8.5 Routing Problems/Travelling Salesman Problems

A salesman, who wishes to travel through his territory visiting various cities, wishes to visit one city only once and wants to come back to the city from where be started and then go to other cities one after other. As he is a single person who has to visit various cities, mere Reduction Theorem Rules or Hungarian approach would not help in arriving at optimum solution always. Hence, a modified solution would be found out to arrive at an optimal solution.

Notes

Example: Solve the following travelling salesman problem with the following matrix.

Travelling Expenses

To

From

	1	**2**	**3**	**4**	**5**
I	-	6	12	6	4
II	6	-	10	5	4
III	8	7	-	11	3
IV	5	4	11	-	5
V	5	2	7	8	-

Solution:

Apply Reduction Theorem Rules

1. Row-wise Reduced Matrix.

To

From

	1	**2**	**3**	**4**	**5**
I	-	2	8	2	0
II	2	-	6	1	0
III	5	4	-	8	0
IV	1	0	7	-	1
V	3	0	5	6	-

2. Column-wise Reduced Matrix.

To

From

	1	**2**	**3**	**4**	**5**
I	-	2	3	1	0
II	1	-	1	0	0
III	4	4	-	7	0
IV	0	0	2	-	1
V	2	0	0	5	
				L_1	L_2

Note: Used Hungarian approach and get the improvised matrices.

3. Modified Matrix – 1.

To

From

	1	**2**	**3**	**4**	**5**
I	-	1	2	1	0
II	0	-	0	0	0
III	3	3	-	7	0
IV	0	0	2	-	0
V	2	0	0	6	0
				L_1	L_2

Notes

4. Modified Matrix – 2.

From \ To	1	2	3	4	5
I	-	0	1	0	0
II	0	-	0	0	1
III	2	2	-	6	0
IV	0	0	2	-	3
V	2	0	0	6	-

The routes chosen for journey are:

Alternative - I		Alternative - II	
From	To	From	To
I	2	I	4
II	4	II	1
III	5	III	5
IV	1	IV	2
V	3	V	3

Note: In both the solutions, there is looping, Hence, it is not an optimum solution to travelling salesman problem. Hence, it is to be improved further.

8.5.1 Steps to Resolve Looping in Travelling Salesman Problem

1. Choose the next minimum uncovered element in the reduced matrix. If there are more than one such numbers, all the possible alternatives are to be tried until an unique optimum solution is found out.
2. Check whether looping is present in modified matrix after trial assignment. Modified matrix having no looping is an optimum solution.

Did u know? The TSP has several applications even in its purest formulation, such as planning, logistics, and the manufacture of microchips. Slightly modified, it appears as a sub-problem in many areas, such as DNA sequencing. In these applications, the concept city represents, for example, customers, soldering points, or DNA fragments, and the concept distance represents travelling times or cost, or a similarity measure between DNA fragments. In many applications, additional constraints such as limited resources or time windows make the problem considerably harder.

Example: A travelling sales man has to visit 5 cities. He wishes to start form a particular city visit each city once and then return to his starting point. The travelling time (in hours) for each city from a particular city is given below:

Notes

To

From		A	B	C	D	E
	a	∞	4	7	3	4
	b	4	∞	6	3	4
	c	7	6	∞	7	5
	d	3	3	7	∞	7
	e	4	4	5	7	∞

What should be the sequence of visit of the salesman, so that the total travelled time is minimum.

Solution:

Application of Reduction Theorem Rules

(1) ***Row-wise Reduced Matrix***

To / From	A	B	C	D	E
a	∞	1	4	0	1
b	1	∞	2	0	1
c	2	1	∞	2	0
d	0	0	4	∞	4
e	0	0	1	3	∞

(2) ***Column-wise Reduced Matrix***

To / From	A	B	C	D	E
a	∞	1	3	0	1
b	1	∞	1	0	1
c	2	1	∞	2	0
d	0	0	3	∞	4
e	0	0	0	3	∞

(i) *Modified Matrix – 1*

To / From	A	B	C	D	E
a	-	0	$\boxed{1}$	0	0
b	0	-	0	$\boxed{0}$	1
c	2	2	-	6	$\boxed{0}$
d	$\boxed{0}$	0	2	-	3
e	2	$\boxed{0}$	0	6	-

Notes

(ii) *Modified Matrix – 2*

From \ To	A	B	C	D	E
a	-	0	1	0	0
b	0	-	0	0	$\boxed{1}$
c	2	2	-	6	0
d	0	0	2	-	3
e	2	0	0	6	-

Note: This alternative cannot be worked out as no unique zero remains in the third row after choosing next minimum element '1' in the II row.

(3) ***Calculation of Travelling Expenses.***

From	To	Travelling Expenses (₹)
a	3	12
b	4	5
c	5	3
d	1	5
e	2	2
	Total Expenses	27

Inference

The minimum travelling expenses with an unique and optimum solution to the above problem works to be ₹ 27.

(4) ***Modified Matrix-1.***

From \ To	A	B	C	D	E
a	∞	$\boxed{0}$	2	0	1
b	0	∞	0	$\boxed{0}$	1
c	1	0	∞	2	$\boxed{0}$
d	$\boxed{0}$	0	3	∞	5
e	0	0	$\boxed{0}$	4	∞

Note: There are three alternative solutions to the above matrix.

(5) (a) Alternative 1 (b) Alternative - 2 (c) Alternative - 3

From	To	From	To	From	To
a	B	a	D	a	D
b	D	b	A	b	C
c	E	c	E	c	E
d	A	d	B	d	A
e	C	e	C	e	B

Note: Observe there is presence of looping in all the three solutions. Hence, they are not the unique solutions to travelling salesman problem. Further, the matrix is to be modified.

Notes

(6) ***Modified Matrix 2.***

To / From	A	B	C	D	E
a	∞	0	2	$\boxed{0}$	$\boxed{1}$
b	0	∞	0	0	1
c	1	$\boxed{0}$	∞	2	0
d	$\boxed{0}$	0	3	∞	5
e	0	0	$\boxed{0}$	4	∞

(7) ***Calculation of Minimum Total Time Required for Travelling.***

From	To	Travelling Time (Hours)
a	E	4
b	D	3
c	B	6
d	A	3
e	C	5
	Total Time	21

Inference

The minimum time required to complete the travel programme with the above unique solution works out to be 21 hours.

Example: A salesman has to visit 4 cities A, B, C, and D. The distance (100 kms) between 4 cities is:

To

From

	A	**B**	**C**	**D**
a	-	4	7	3
b	4	-	6	3
c	7	6	-	7
d	3	3	7	-

If the salesman starts from city 'A' and comes back to city 'A', which route should he select so that total distance travelled by him is minimum?

Solution:

Application of Reduction Theorem Rules

(1) Row-wise Reduced Matrix.

To / From	A	B	C	D
a	-	1	4	0
b	1	-	3	0
c	1	0	-	1
d	0	0	7	-

Notes

(2) Column-wise Reduced Matrix.

To / From	A	B	C	D
a	-	1	1	$\boxed{0}$
b	1	-	$\boxed{0}$	0
c	1	$\boxed{0}$	-	1
d	$\boxed{0}$	0	4	-

(3) The Route Chosen for Journey are:

From	To	Journey Time
a	D	3
b	C	6
c	B	6
d	A	3
Total time		18

Note: There is looping in the above solution. Hence, it is not a unique solution to the travelling salesman problem. Hence, requires further modification.

(4) Modified Matrix - 1

	A	B	C	D
a	-	1	1	$\boxed{0}$
b	1	-	$\boxed{0}$	0
c	$\boxed{1}$	0	-	1
d	0	$\boxed{0}$	4	-

(5) Calculation of Minimum Total Time Required to complete the journey.

From	To	Journey Time
a	D	3
b	C	6
c	A	7
d	B	3
	Total time	19

Inference

The unique and optimum solution shows that the minimum time required is 19 hours.

Task For a salesman who has to visit n cities, which of the following are the ways of his tour plan

a. n! b. (n+1)!

c. (n-1)! d. n

Self Assessment

Notes

State true or false:

7. For a salesman to visit n cities, there are (n+1)! ways to plan his tour.
8. An essential condition in a travelling salesman problem is that a salesman, who wishes to travel through his territory visiting various cities, wishes to visit one city only once.
9. Mere Reduction is not the solution to travelling salesman problem; hence the solution is to find an optimal route that could achieve the objective of the salesman.

8.6 Summary

- Assignment problem is one of the special cases of transportation problems. The goal of the assignment problem is to minimize the cost or time of completing a number of jobs by a number of persons. An important characteristic of the assignment problem is the number of sources is equal to the number of destinations. It is explained in the following way.
    1. Only one job is assigned to person.
    2. Each person is assigned with exactly one job.
- Assignment problem can have various variants such as maximization assignment problem, unbalanced assignment problem, multiple optimal solutions assignment problems and travelling salesman problem.
- The assignment problem where the number of persons is not equal to the number of jobs is called an unbalanced assignment problem. A dummy variable, either for a person or job (as it required) is introduced with zero cost or time to make it a balanced one.
- While making an assignment in the reduced assignment matrix, it is possible to have two or more ways to strike off a certain number or zeroes. Such a situation indicates multiple optimal solutions with the same optimal value of objective function.
- In maximization assignment problems, the objective is to maximize profit, revenue, etc. Such problems can be solved by converting the given maximization problem into a minimization problem.
- In a travelling salesman problem is a special type of Assignment problem. In this given a number of cities and the costs of travelling from one to the other, it is required to determine the cheapest route that visits each city once and then returns to the initial city.

8.7 Keywords

An Infeasible Assignment: Infeasible assignment occurs when a person is incapable of doing certain job or a specific job cannot be performed on a particular machine. These restrictions should be taken in to account when finding the solutions for the assignment problem to avoid infeasible assignment.

Balanced Assignment Problem: This is an assignment where the number of persons is equal to the number of jobs.

Dummy Job/ Person: Dummy job or person is an imaginary job or person with zero cost or time introduced in the unbalanced assignment problem to make it balanced one.

Unbalanced Assignment Problem: This is the case of assignment problem where the number of persons is not equal to the number of jobs. A dummy variable, either for a person or job (as it required) is introduced with zero cost or time to make it a balanced one.

Notes

Travelling Salesman Problem: The problem in combinatorial optimization in which, given a number of cities and the costs of travelling from one to the other, it is required to determine the cheapest route that visits each city once and then returns to the initial city.

8.8 Review Questions

1. Discuss the variations in assignment problem.
2. Can there be multiple optimal solutions to an assignment problem? How would you identify the existence of multiple solutions if any?
3. How would you deal with the assignment problems, where the objective function is to be maximized?
4. What is an unbalanced assignment problem? Explain with the help of an example.
5. How is the Hungarian method applied for obtaining a solution if the matrix is rectangular?

Problems

6. A manufacturer of garments plans to add 4 regional warehouses to meet increased demand. The following bids in lakhs of rupees have been for construction of the warehouses.

Contractor	Warehouse			
	A	B	C	D
1	30	27	31	39
2	28	18	28	37
3	33	17	29	41
4	27	18	30	43
5	40	20	27	36

Explain why each warehouse contract cannot simply be given to the contractor who bids cheapest. How will you then determine the optimal contract?

7. 5 operators have to be assigned to 5 machines. The assignment costs are given in the table.

Operator	Machine				
	I	II	III	IV	V
A	5	5	-	2	6
B	7	4	2	3	4
C	9	3	5	-	3
D	7	2	6	7	2
E	6	3	7	9	1

Operator A cannot operate machine II and operator C cannot operate machine IV. Find the optimal assignment schedule.

Notes

8. Determine an optimum assignment schedule for the following assignment problem. The cost matrix is –

Job	Machine					
	1	**2**	**3**	**4**	**5**	**6**
A	11	17	8	16	20	15
B	9	7	12	6	15	13
C	13	16	15	12	16	8
D	21	24	17	28	26	15
E	14	10	12	11	15	6

If job C cannot be assigned to machine 6, will the optimum solution change?

9. A company has 6 jobs to be processed by 6 machines. The following table gives the return in rupees when the ith job is assigned to the jth machine (I, J = 1…….6). How should the jobs be assigned to the mechanics so as to maximize the overall return?

Machine	Jobs					
	1	**2**	**3**	**4**	**5**	**6**
1	9	22	58	11	19	27
2	43	78	72	50	63	48
3	41	28	91	37	45	33
4	74	42	27	49	39	32
5	36	11	57	22	25	18
6	13	56	53	31	17	28

10. A company is faced with the problem of assigning 4 machines to 6 different jobs (one machine to one job only). The profits are:

Job	Machine			
	A	**B**	**C**	**D**
1	3	6	2	6
2	7	1	4	4
3	3	8	5	8
4	6	4	3	7
5	5	2	4	3
6	5	7	6	4

11. A process can be carried out an anyone of 6 machines. The average time taken by any operator on any specific machine is tabulated in the given matrix. It is proposed to buy a new machine to replace one of the existing ones for operations. To carry out the process on this machine average times have been estimated and entered in the matrix. Is it advantageous at this stage to use the new machine? If so, which of the original machines should be replaced and how should the operators be allocated?

Notes

Machine Operators	Avg. time on machines						
	1	2	3	4	5	6	New
A	10	12	8	10	8	12	11
B	9	10	8	7	8	9	10
C	8	7	8	8	8	6	8
D	12	13	14	14	15	14	11
E	9	9	9	8	8	10	9
F	7	8	9	9	9	8	8

12. A section head has five stenotypists and five jobs to complete. The stenos differ in their efficiency and the jobs differ in their intrinsic complexity. The estimate of the time (in hours) each steno would take to perform the task is given in the effectiveness matrix below. How should tasks be allocated, one to a person, so as to maximize the total time taken to complete all the jobs?

Tasks	Steno-typists				
	A	B	C	D	E
I	10	17	15	11	14
II	18	22	14	16	12
III	29	20	12	18	22
IV	14	19	7	17	16
V	9	17	14	12	15

13. A tourist car rental agency has a surplus car in each of the cities A, B, C, D, E and F, and a deficit of one car in each of the cities AA, BB, CC, DD, EE, and FF. The distances between cities with a surplus car and cities with a deficit car are given in the following matrix. How should the cars be dispatched so as to maximize the total distance covered?

From	To					
	AA	BB	CC	DD	EE	FF
A	28	39	47	51	36	42
B	44	49	38	25	29	32
C	59	50	31	31	21	28
D	48	64	80	49	26	38
E	40	71	38	51	50	29
F	39	39	80	59	40	71

14. A domestic airline which operates seven days a week has a time table between two airports as shown below. Crew must have a minimum layover of 2 hours between any two flights. Obtain an optimal pairing of flights to minimize layover time away from HQ. For any given pairing, the crew will be based at the city that results in the smaller layover.

Chennai-Bangalore			Bangalore-Chennai		
Ft. no.	Departure	Arrival	Ft. no.	Departure	Arrival
185	6.30 a.m	7.00 a.m	186	8.00 a.m	8.30 a.m
255	8.00 a.m	8.30 a.m	256	10.00 a.m	10.00 a.m
275	2.00 a.m	2.30 a.m	286	7.00 a.m	1.30 a.m
285	6.00 p.m	6.30 a.m	286	7.00 a.m	7.30 a.m

Notes

15. Carew's Machine shop has four jobs of which three jobs have to be done. Each job can be assigned to one and only one machine. The cost (in ₹) of each on each machine is given below. What are the job assignments which will minimize the total cost?

Job	Machine			
	m_1	m_2	m_3	m_4
J_1	145	160	170	180
J_2	220	132	242	147
J_3	225	237	147	255

16. In response to an advertisement for sale of old vehicles, the bank has received tender bids from four parties for four vehicles. Although anyone can make bids on all four vehicles, the bank has decided to accept only one bid per party. The bids (in ₹) made by the four parties are given below.

Party	Vehicles			
	Premier	Ambassador	Maruti	Jeep
Patel	75,000	87,500	72,500	68,000
Shah	77,500	85,000	73,000	66,000
Vohra	74,000	86,000	72,000	80,000
Trivedi	75,000	88,500	72,000	80,000

Whom should the bank award the vehicles in order to maximize the total revenue?

17. A large consumer marketing company is planning to cover 5 more states for sales promotion. However, the company could recruit four new experienced salesmen. On analyzing the new salesmen's past experience in conjunction with a performance test which was given to them, the company assigned performance ratings to them for each of the state. These ratings are given below. As the company's policy is to attach one salesman to one state, with four salesmen it would not be possible to go for all the five states. Therefore, company would like to know which four states have to be selected in order to maximize the total performance.

Salesman	States				
	Punjab	Haryana	H.P.	Orissa	Bihar
A	46	42	44	39	45
B	45	45	44	42	47
C	47	46	41	43	48
D	40	47	45	41	47

18. A company has 6 machines and five skilled operators to operate them. The performance (in terms of number of units produced in an hour) of these five operators on these six machines is known to the company and the same is given in following table:

Notes

Operators	Machines					
	1	2	3	4	5	6
A	25	20	30	35	40	60
B	50	50	65	70	70	90
C	10	15	20	20	30	50
D	60	70	80	90	90	100
E	20	30	30	45	60	60

How would you assign the operators, one to a machine, so as to maximize the output?

Answers: Self Assessment

1. (d)
2. (c)
3. (a)
4. Square
5. Zero
6. Number of jobs and resources
7. False
8. True
9. True

8.9 Further Readings

Books

J.K. Sharma, *Operations Research, Theory and Applications,* MacMillan India Ltd.

Kanti Swarup, P.K Gupta & Manmohan, *Operations Research*, Sultan Chand Publications, New Delhi

Michael W. Carter, Camille C. Price, *Operations Research: A Practical Introduction,* CRC Press, 2001

Paul A. Jensen, Jonathan F. Bard, *Operations Research Models and Methods,* John Wiley and Sons, 2003

Richard Bronson, Govindasami Naadimuthu, *Schaum's Outline of Theory and Problems of Operations Research,* McGraw-Hill Professional; 1997

Online links

www.utdallas.edu/

http://www.usna.edu/

http://businessmanagementcourses.org/

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Unit 9: Game Theory

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Objectives

After studying this unit, you will be able to:

- Understand basic terminologies used in game theory
- Familiarize with various methods solving games.
- Analyze limitations of games in competitive situations

Introduction

Game Theory was originally developed by John Von Neumann in 1928. The mathematical relationship between game theory and linear programming was also initially recognized by Von Newmann. However, George B. Dantiz was the person to apply the simplex method successfully to solve a game theory problem. Von Neumann's book entitled "*Theory and Practice of Games and Economic Behaviour*", which he authored with Morgenstern, is considered as a pioneer work by the experts all over. It had a great impact on the development of Linear Programming and wall's Statistical Decision Theory.

Business decisions in a competitive situation do not depend on the decisions of the organization alone but on the interaction between the decisions of the organization and those of the competitors. Each firm tries to select and execute its strategies and aims to maximize its gains at the cost of its opponents. Similarly, a competitor too rises to select the best of his strategies to counteract his opponents again. Game theory deals with problems where actions and interactions of competing firms give rise to conditions of business conflict. In other words, Game Theory is a body of knowledge which is concerned with the study of decision-making in situations where two or more rational opponents are involved under conditions of competition and conflicting interest. It deals with human processes which an individual, a group, a formal or informal organization or a society, is not in complete control of the other decision-making units, the opponents, and is addressed to problems involving conflict, co-operation or both at various levels.

Notes

Some of the competitive situations in economic, social, political or military activities are:

(1) Firms trying to snatch each other's market share.

(2) Military attacks.

(3) Selection of best advertising media.

Game must be thought in a broad sense not as a kind of sport like chess or bridge but competitive situation, a kind of conflict in which one must win and the other must lose. The following are some of the fields of application of game theory:

(a) In a competitive market, sometimes companies wage a price war. What should be the bid to win major government contract in the face of competition from several contractors.

(b) An equipment dealer and a customer may be at Cross purposes regarding price but they would both want to close a mutual advantageous deal. Similarly, in a collective bargaining process, the trade union and the management of a company share the objective of striking at a mutually advantageous deal and keep the company operations going.

(c) Suppose a firm wants to introduce a new product in the market, to get a bigger share in the market, the marketing manager of the firm would be interested to know the best possible strategies of a competitor who is also trying to introduce product with different strategies e.g., price reduction, better quality, etc.

Notes The models in the theory of games can be classified depending upon following factors:

1. Number of players
2. Sum of gains and losses
3. Strategy

9.1 Terminologies in Game Theory

The participants to the game who act as decision-makers are called players. In a game two or more participants may be in the conflict. The former of these is called a *two person game* and the latter one is known as a *person game*. Where it does not necessarily imply that in its play exactly n people would be involved, but rather that the participants can be classified into n mutually exclusive categories and members of each of the categories have identical interest.

A finite or infinite number of possible courses of action available to a player are called *strategies.*

Example: Let x and y be two manufacturers and x is faced with a problem of deciding whether it is worthwhile to reduce the price of the product to counteract competition. He has two strategies:

1. reduce the price, and
2. maintain the price. Then y too has the same strategies to counter act x's strategies.

Notes

Play

A play occurs when each player selects one of his available strategies. Two basic assumptions in a play are:

(a) The choices of courses of action by players are made simultaneously.

(b) No player knows the choice of his opponents until he has decided on his own.

Outcome

Every combination of strategies of players determines an outcome called pay-off, where pay-off is nothing but a gain to a player. A loss is considered as a negative gain.

Pay-off Matrix

The gains resulting from a game is presented in the form of a table called " pay-off matrix". A pay-off matrix comprises n rows and m columns. Where n and m indicate the number of strategies of first player and second player respectively. The pay-offs of each combination of the strategies of players are placed as elements of matrix. A positive element shows the gain to the first player (i.e., payment from II to I) and negative entry indicates the loss to the I player (i.e., payment from I to II).

For instance, consider the following pay-off matrix:

	1	2	3	4	5
1	4	8	-2	6	4
2	3	6	5	3	2
3	2	-9	1	7	10

If the player chooses the first strategy and the II player uses second strategy, then the I player gains 8 units and the II player pays 8 units and similarly, if the 1 player chooses the third strategy and second player uses II strategy, then the 1 player loses 9 units and pays it to the II player and II player gains 9 units.

Strategies are classified into two types, namely, Pure strategy and Mixed Strategy.

1. A *pure Strategy* is a decision of the player to always select the same strategy.
2. A *Mixed Strategy* is a decision of the player to select more than one strategy with fixed probabilities. A mixed strategy is advantageous since the opponent is always kept guessing.

Value of the Game

The value of the game is the "expected gain to a player" if he and his opponent use their best strategies.

Saddle Point

A saddle point in a pay-off matrix corresponds to that element of the matrix which represents the 'Maxmin' value of a player and Minimax value of his opponent.

For this we find Maximum element of each column and then find the Minimum value of column Maxima known as Minimax. Similarly, we identify minimum element of each row and then find the Maximum of those entries known as Maximin. If Minimax = Maximum of those entries

Notes

known as Maximin. If Minimax = Maximin, then saddle point exists and the value of the game is equal to Minimax to Maximin.

If Minimax $\neq$ Maximin, then no saddle point exists.

If Minimax = Maximin, then the pure strategies are called optimum strategies.

Usually Maximin $\leq$ value of the game $\leq$ Minimax. If Maximin = Minimax = 0, the game is fair. If Maximin = Minimax, the game is strictly determinable. The value of the game is the average pay-off that would suit the game was played over and over again.

Self Assessment

Give one word for the following:

1. The list of all possible actions that a player takes for every payoff.
2. A strategy that always involves selecting a particular course of action with the probability of 1.
3. The strategy to choose at least two courses of action with fixed probabilities.
4. A situation where both the players are facing pure strategies.
5. The "expected gain to a player" if the player and his opponent use their best strategies.

9.2 Two Person Zero Sum Game

In a game of two persons if the algebraic sum of the gains of both the players after the game is zero, then it is called 2-person-zero-sum game.

Assumptions

1. There are 2 players having conflicting interests.
2. Each player has a finite number of strategies.
3. Each strategy selected by a player results into a certain pay-off and algebraic sum of these pay-offs to both players is zero.

Two-person-zero-sum game with saddle point are called pure strategy games and two-person-zero-sum game without saddle point are called mixed strategy games.

Let A and B be any 2 firms in an area have been selling a product competing for a larger share of the market. Let us assume that these firms are considering the same three strategies in a bid to gain the share in the market: Low advertising, high advertising and quality improvement and let these firms can employ only one of the strategies at a time. Under these conditions, there are 3 x 3 combinations of the moves possible and the corresponding pay-off is given below. The strategies of low advertising, high advertising and quality improvement is marked as a_1, a_2 and a_3 for the firm A and b_1, b_2 and b_3 for the firm B.

	B's Strategy				
		b_1	b_2	b_3	Row Min.
A's Strategy	a_1	12	-8	-2	-8
	a_2	6	7	3	3
	a_3	-10	-6	2	-10
Col. Max.		12	7	3	

Notes

Maximin = Maximum of Row Minimum =3

Minimax = Minimum of Column Maximum = 3

Therefore Saddle point = (a_2, b_3), and

Value of the game = 3

Thus pay-off matrix is drawn from as point of view. A positive pay-off indicates that firm A has gained the market share at the expense of firm B and negative value indicates B's gain at A's expense. The problem now is to determine the best strategy for A and B. With the assumption that each one is not aware of the move the other is likely to take reference to the pay off matrix if firm A employs strategy a_1, then firm B employs strategy b_2 in order to maximize its gain. Similarly, if A's strategies are a_2 and a_3, then B's strategies are b_3 and b_1 respectively. Now, firm A would like to make the best use of the situation by choosing the maximum of these minimal pay-offs. Since the minimal pay-offs corresponding to a_1, a_2 and a_3 are respectively -8, 3 and -10, firm A would select a_2 as its strategy. The decision rule here is *Maximin strategy.* Similarly, if firm chooses b_1, then A will prefer a_1 and if B uses b_2 and b_3 then firm A uses the strategy a_2. To minimize the advantage occurring to A, firm B would select a strategy a_2. To minimize the advantage occurring to A, firm B would select strategy a_2. To minimize the advantage occurring to A, firm B would select a strategy that would yield the least advantage to its competitor, i.e., b_3. The decision rule here is Minimax strategy.

Here, Minimax value = Maximin value = 3, which the value of the game and corresponds to the saddle point.

Saddle point can be easily obtained for 2-person pure strategy games. We shall deal with *2-person mixed strategy games.*

Here, the players play more than one strategy and no saddle point exists. To determine the optimal strategies, the analyst needs to evaluate the probabilities (the proportion of time for which each strategy is played). For doing so we have 3 methods namely:

1. Algebraic Method
2. Iterative Method for Approximate Solution
3. Linear Programming Method.

9.2.1 Algebraic Method

Let A and B be any two players with the following pay-off matrix; a_1, a_2 and b_1, b_2 denote the strategies of A and B respectively. Let Pij denote the elements of pay-off matrix i, j = 1,2

Player B

Player A

Strategies	b_1	b_2
a_1	P_{11}	P_{12}
a_2	P_{21}	P_{22}

Player A has only 2 strategies namely, a_1 and a_2. If probability that he chooses a_1 is x then probability that he chooses a_2 is 1 – x. Similarly, if probability that player B chooses b_1 and b_2 is y and 1 – y respectively. Let us consider the expected gain which is the weighted average of the possible outcomes and is the product of payoff and the probabilities of the strategies. If player B plays b_1 throughout, then the gain to A is equal to

$$xP_{11} + (1-x)P_{21} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Notes

If B play b_2 throughout, then the gain A is equal to

$$xP_{12} + (1-x)\,P_{22} \qquad \text{..... (2)}$$

From (1) and (2)

$$xP_{11} + (1-x)P_{21} = xP_{12} + (1-x)\,P_{22}$$

$$xP_{11} - xP_{12} + (1-x)P_{21} - (1-x)P_{22} = 0$$

$$x(P_{11} - P_{12}) + 1(P_{21} - P_{22}) - x\,(P_{21} - P_{22}) = 0$$

$$x[P_{11} - P_{12}) - (P_{21} - P_{22})] = -(P_{21} - P_{22})$$

$$x[(P_{11} - P_{21}) + (P_{22} - P_{21})] = (P_{22} - P_{21})$$

Therefore

$$x = \frac{(P_{22} - P_{21})}{(P_{11} - P_{12}) + (P_{22} - P_{21})} \qquad \text{..... (3)}$$

A similar argument holds good for B whose best strategy is

$$y = \frac{(P_{22} - P_{12})}{(P_{11} - P_{12}) + (P_{22} - P_{21})}$$

The value of the game i.e., gain to A from B can be obtained by substituting for x in (1) which on substitution and rearrangement becomes

Gain = $\dfrac{(P_{11}P_{22} - P_{21}P_{12})}{(P_{11} - P_{12}) + (P_{22} - P_{21})}$; i.e. $|P|$ (Sum of row differences)

Example: The pay-off matrix is

	B		Row Min.
A	1 5	4 3	1 3
Col. Max.	5	4	

Minimax = 4 and Maximin = 3

Here Minimax ≠ Maximin. Hence no saddle point exists. Let x and 1–x be the probability of A playing a_1, and a_2 and y and 1 – y be the probability of B playing b_1 and b_2 respectively.

We have

$$x = \frac{(P_{22} - P_{21})}{(P_{11} - P_{12}) + (P_{11} - P_{12})}$$

$$= \frac{(3-5)}{(1-4) + (3-5)}$$

$$= \frac{-2}{-5} = \frac{-2}{-3 + (-2)}$$

Therefore $= \dfrac{2}{5}$ = Hence, $1 - x = \dfrac{3}{5}$

Notes

Similarly,

$$y = \frac{(P_{22} - P_{12})}{(P_{11} - P_{21}) + (P_{22} - P_{12})}$$

$$= \frac{(3-4)}{(1-5)+(3-4)}$$

$$= \frac{-1}{-4+-1}$$

$$= \frac{-1}{-5}$$

Therefore, $y = \frac{1}{5}$

$$1 - y = \frac{4}{5}$$

Therefore Optimum mixed strategies for player B are to play first column 1/5 of the time and second column 4/5 of the time and that for A are to play first row 2/5 of the time and second row 3/5 of the time value of the game

$$V = \frac{P_{11}P_{22} - P_{12}P_{21}}{(P_{11} - P_{12}) + (P_{22} - P_{21})}$$

$$= \frac{1 \times 3 - 5 \times 4}{(P_{11} - P_{12}) + (P_{22} - P_{21})}$$

$$= \frac{3-20}{-3-2}$$

$$= \frac{-17}{-15}$$

$$= 3.4$$

Iterative Method for Approximate Solution

In many practical problems, exact optimal solution of the game is not required. It is sufficient to find out an approximate solution which gives an average gain, close to the value of the game. This is one of such method based on the principle that "Two players are supposed to play the game iteratively and at each play the players choose the strategy which is best to himself or say worse to opponent, in view of which the opponent has done that iteration."

Suppose A (maximizing player) starts the game by choosing his strategy Ai arbitrarily Vi = 1 to n, then B (minimizing player) chooses that strategy Bj (Vj = 1 to m) which is best to himself or worst for A. Now for this strategy of B, A chooses his strategy Ar of which maximizes his average gain. Now B responds to the strategies Ak and Ar by his strategy Bs which minimizes the average loss to him. For this, B adds both strategies of A and then chooses his strategy for B which corresponds to the least element. Now A adds both the strategies of Bj and chooses his strategy which corresponds to the maximum element in B's strategy. Ultimately at any iteration a mixed strategy can be obtained by dividing the number of times the respective pure strategies used by the total number of iterations up to that stage. The method is slow and many iterations are involved but this is useful for large games.

Notes

Linear Programming Method

When 2 person game has more than two options we can make use of linear programming method to establish the opposing player's strategies. Let A and B be two players with m and n options respectively then the payoff matrix is given by,

Player B

Player A		b_1	b_2……. b_n
	A_1	a_{11}	a_{12}……a_{ln}
	a_2	a_{21}	a_{22} ……a_{2n}
	.	.	
	.	.	
	.	.	
	.	.	
	a_m	a_{ml}	a_{m2}……a_{mn}

The problem is to find the fixed optimal strategies. Let x_1, x_2… X_m and y_1, y_2……..Yn denote the probability that the player A chooses the strategies a_1, a_2…….a_m and B chooses the strategies b_1, b_2……..b_n such that $\sum_{i=1}^{m} x_1 = 1$ and $\sum_{i=1}^{n} y_1 = 1$ and if the value of the game is V, then it is possible to formulate LPP as below.

Maximise V = Expected payoff

$$\left.\begin{array}{l} \text{Subject to } a_{11}X_1 + a_{21}X_2 + \ldots\ldots\ldots.. a_{m1}X_m \geq V \\ a_{12}X_1 + a_{22}X_2 + \ldots\ldots\ldots.. am_2X_m \geq V \\ . \\ . \\ . \\ . \\ . \\ a_{ln}X_1 + a_{2n}X_2 + \ldots\ldots\ldots\ldots.. + a_{mn}X_m \geq V \\ X_1X_2X_3 \ldots\ldots\ldots\ldots\ldots\ldots.. X_m \geq 0 \end{array}\right\} \quad (1)$$

Objective function can be expressed as,

$$\text{Max. } V = \text{Min. } \frac{1}{V} \ (V > 0)$$

$$= \text{Min. } \frac{(x_1 + x_2 + \ldots\ldots\ldots\ldots + x_m)}{V}$$

$$= (x_1^1 + x_2^1 \ldots\ldots\ldots\ldots\ldots + x_m^1) \text{ where } x_1^1 = \frac{x1}{V} \ V = 1 \text{ to m.}$$

Notes

Constraints are,

$$\left.\begin{array}{l} a_{11}X_1^1 + a_{21}X_2^1 + \ldots\ldots\ldots\ldots\ldots + a_{m1}X^1_m \geq 1 \\ a_{12}X_1^1 + a_{22}X_2^1 + \ldots\ldots\ldots\ldots\ldots + a_{m2}X^1_m \geq 1 \\ . \\ . \\ . \\ . \\ . \\ a_{ln}\, X_1^1 + a_{2n}X_2^1 + \ldots\ldots\ldots\ldots\ldots + a_{mn}X^1_m \geq 1 \\ V\ x_i^1 \geq 0,;\ \ i = 1 \text{ to } m \end{array}\right\} \quad (2)$$

B's strategies can be written as,

Min. v = Expected loss

$$\text{Subject to } \left.\begin{array}{l} a_{11}Y_1 + a_{12}Y_2 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{ln}Y_n \leq V \\ a_{21}Y_1 + a_{22}Y_2 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{2n}Y_n \leq V \\ . \\ . \\ . \\ . \\ a_{m1}Y_1 + a_{m2}Y_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{mn}Y_n \leq V \\ \text{If } Y_i^1 = \dfrac{Yi}{V} \qquad\qquad V\ i = 1 \text{ to } n, \end{array}\right\} \quad (3)$$

then the constraints can be written as,

$$\left.\begin{array}{l} a_{11}Y_1^1 + a_{12}Y_2^1 + \ldots\ldots\ldots\ldots + a_{ln}Y_n^1 \leq 1 \\ a_{21}Y_1^1 + a_{22}Y_2^1 + \ldots\ldots\ldots\ldots + a_{2n}Y_n^1 \leq 1 \\ . \\ . \\ a_{m1}Y_1^1 + a_{m2}Y_2^1 + \ldots\ldots\ldots + a_{mn}Y_n^1 \leq 1 \\ V\ yi \geq 0;\ V\ i = 1 \text{ to } n. \end{array}\right\} \quad (4)$$

The set of inequalities (2) is dual of set (4). Solving (4) by simplex method optimal solution is obtained.

Example: Formulate the following game as an LPP and obtain its solution:

A's Strategy	B's Strategy			
		B_1	B_2	B_3
	a_1	8	9	3
	a_2	2	5	6
	a_3	4	1	7

Notes

The given problem can be formulated as an LPP from A's and B's point of view as follows:

Let x_1, x_2 and x_3 be the probabilities with which A chooses respectively the strategies a_1, a_2 and a_3 and y_1, y_2 and y_3 be the probabilities in respect of B choosing b_1, b_2 and b_3 respectively. From A's point of view we

$$\text{Minimise } \left(\frac{1}{V}\right) = x_1 + x_2 + x_3$$

Subject to
$$8x_1 + 2x_2 + 4x_3 \geq 1$$
$$9x_1 + 5x_2 + x_3 \geq 1$$
$$3x_1 + 6x_2 + 7x_3 \geq 1$$
and
$$x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$
$$\text{Where, } X_i = \frac{X_i}{V}$$

From B's point of view, we have

Maximise
$$\frac{1}{V} = y_1 + y_2 + y_3$$

Subject to
$$8y_1 + 9y_2 + 3y_3 \leq 1$$
$$2y_1 + 5y_2 + 6y_3 \leq 1$$
$$4y_1 + y_2 + 7y_3 \leq 1$$
$$y_1, y_2, y_3 \geq 0; \text{ and } Y_1 = \frac{Y_i}{V}$$

To calculate the required values we can solve either of these LPPs and read solution to the other from it as each one is the dual of the other. We shall solve the game from B's point of view.

Introducing slack variables the objective function can be written as,

Maximise $1 = y_1 + y_2 + y_3 + OS_1 + OS_2 + OS_3$

Subject to
$$8y_1 + 9y_2 = 3y_3 + S_1 = 1$$
$$2y_1 + 5y_2 + 6y_3 + S_2 = 1$$
$$4y_1 + y_2 + 7y_3 + S_3 = 1$$
$$y_1, y_2, y_3 \geq 0; s_1, s_2, s_3 \geq 0$$

Simplex Table 1

BV	C_B	X_B	Y_1	Y_2	Y_3	Min. Ratio	S_1	S_2	S_3
S_1	0	1	8	9	3	1/8 = 0.125	1	0	0
S_2	0	1	2	5	6	½ = 0.50	0	1	0
S_3	0	1	4	1	7	¼ = 0.25	0	0	1
ZB = 0		Z_j C_j	0 1	0 1	0 1		0 0	0 0	0 0
		$Z_j - C_j$	–1	–1	–1		0	0	0

Notes

Simplex Table 2

BV	C_B	X_B	Y_1	Y_2	Y_3	Min. Ratio	S_1	S_2	S_3
Y_1		1/8	1	9/8	3/8	$\frac{1/8}{3/8} = \frac{1}{3} = 0.33$	$\frac{1}{8}$	0	0
S_2	0	¾	0	11/4	21/4	$\frac{3/4}{21/4} = \frac{3}{21} = 0.1418$	–¼	1	0
S_3	0	½	0	–7/2	11/2	$\frac{1/2}{11/2} = \frac{1}{11} = 0.0909$	–½	0	1
$Z_B = 1/8$		Zj Cj	1 1	9/8 1	3/8 1	– –	1/8 0	0 0	0 0
		Zj – Cj	0	–1/8	–5/8	–	+ 1/8	0	0

Simplex Table 2

BV	C_B	X_B	Y_1	Y_2	Y_3	Min. Ratio	S_1	S_2	S_3
Y_1	1	$\frac{1}{11}$	1	$\frac{15}{11}$	0	$\frac{1/11}{15/11} = \frac{1}{15} = 0.06$	$\frac{7}{44}$	0	$\frac{-3}{44}$
S_2	0	$\frac{3}{11}$	0	$\frac{67}{11}$	0	$\frac{3/11}{67/11} = 0.044$	$\frac{5}{22}$	1	$\frac{-21}{22}$
Y_3	1	$\frac{1}{11}$	0	$\frac{7}{11}$	1	–	$\frac{1}{11}$	0	$\frac{2}{11}$
$Z_B = 2/11$		Z_j C_j	1 1	$\frac{8}{11}$	1 1	– –	$\frac{3}{44}$ 0	0 0	$\frac{5}{44}$ 0
		$Z_j - C_j$	0	$\frac{-3}{11}$	0	–	$\frac{3}{44}$	0	$\frac{5}{44}$

Simplex Table 2

BV	C_B	X_B	Y_1	Y_2	Y_3	Min. Ratio	S_1	S_2	S_3
Y_1	1	2/67	1	0	0	–	29/268	–15/67	39/268
Y_2	1	3/67	0	1	0	–	5/134	11/67	–21/134
Y_3	1	8/67	0	0	1	–	–9/134	7/67	11/134
		Z_j C_j	1 1	1 1	1 1	– –	21/268 0	12/268 0	19/268 0
		$Z_j - C_j$	0	0	0	–	V $Z_j - C_j$ 21/268	0 12/268	J = 1 to 312 19/268

Substituting for Y_1, Y_2, Y_3 in the objective function.

We have Max. = $\frac{1}{V} + \frac{2}{67} + \frac{3}{67} = \frac{8}{67}$

Therefore value of the game = V = $\frac{13}{67}$ = 5.15

Notes

$$Y_i = \frac{Y_i}{V} \qquad V\ i = 1 \text{ to } 3$$

Therefore $y_i = Vy_i$ V I = 1 to 3

Therefore

$$y_1 = Vy_1 = \frac{67}{13} \times \frac{2}{67} \times \frac{3}{13}$$

$$y_2 = Vy_2 = \frac{67}{13} \times \frac{3}{67} \times \frac{3}{13}$$

$$y_3 = Vy_3 = \frac{67}{13} \times \frac{8}{67} \times \frac{8}{13}$$

We can read the values of the dual variables X_1 X_2 and X_3 for $(Z_j - C_j)$ row of slack variables in Table 4 which are $\frac{21}{268}, \frac{12}{268}$ and $\frac{19}{268}$ respectively.

$$\text{Min. } \frac{1}{V} = \frac{21}{268} + \frac{12}{268} + \frac{19}{268} = \frac{52}{268} = \frac{13}{67}$$

Therefore $V = \frac{67}{13}$

Hence,

$$x_1 = \frac{21}{268} \times \frac{67}{13} = \frac{21}{52}$$

$$x_2 = Vx_2 = \frac{67}{13} \times \frac{12}{268} = \frac{12}{52} = \frac{3}{13}$$

$$x_3 = Vx_3 = \frac{67}{13} \times \frac{14}{268} = \frac{19}{52}$$

Therefore the optimal strategy for A is $\frac{21}{52}, \frac{12}{52}$ and $\frac{19}{52}$ and for B is 2, 3, and $\frac{2}{13}, \frac{3}{13}$ and $\frac{8}{13}$ and the value of the game is B = $\frac{67}{13}$

Self Assessment

State true or false:

6. Two-person-zero-sum game without saddle point is called pure strategy games.
7. One of the important assumptions in a two person zero sum game is that each player has a finite number of strategies.
8. Two-person-zero-sum game with saddle point is called mixed strategy games.

9.3 Pure Strategies: Game with Saddle Point

The aim of the game is to determine how the players must select their respective strategies such that the payoff is optimized. This decision-making is referred to as the **minimax-maximin principle** to obtain the best possible selection of a strategy for the players.

In a payoff matrix, the minimum value in each row represents the minimum gain for player A. Player A will select the strategy that gives him the maximum gain among the row minimum

Notes

values. The selection of strategy by player A is based on maximin principle. Similarly, the same payoff is a loss for player B. The maximum value in each column represents the maximum loss for Player B. Player B will select the strategy that gives him the minimum loss among the column maximum values. The selection of strategy by player B is based on minimax principle. If the maximin value is equal to minimax value, the game has a saddle point (i.e., equilibrium point). Thus the strategy selected by player A and player B are optimal.

Example: Consider the example to solve the game whose payoff matrix is given in Table below:

$$\begin{array}{cc} & \textbf{Player B} \\ & \begin{array}{cc} \textbf{1} & \textbf{2} \end{array} \\ \textbf{Player A} & \begin{array}{c} \textbf{1} \\ \\ \textbf{2} \end{array} \begin{pmatrix} 1 & 3 \\ -1 & 6 \end{pmatrix} \end{array}$$

Solution: The game is worked out using minimax procedure. Find the smallest value in each row and select the largest value of these values. Next, find the largest value in each column and select the smallest of these numbers. The procedure is shown in Table below.

Minimax Procedure

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{ccc} 1 & 2 & \text{Row Min} \end{array} \\ \text{Player A} & \begin{array}{c} 1 \\ \\ 2 \end{array} \begin{pmatrix} 1 & 3 \\ -1 & 6 \end{pmatrix} \begin{array}{c} 1 \\ -1 \end{array} \\ \text{Col Max} & \begin{array}{cc} 1 & 6 \end{array} \end{array}$$

If Maximum value in row is equal to the minimum value in column, then saddle point exists.

$$\text{Max Min} = \text{Min Max}$$

$$1 = 1$$

Therefore, there is a saddle point.

The strategies are,

Player A plays Strategy A_1, $(A \rightarrow A_1)$.

Player B plays Strategy B_1, $(B \rightarrow B_1)$.

Value of game = 1.

Example: Solve the game with the payoff matrix for player A as given in Table below.

Game Problem

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Player A} & \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} \begin{pmatrix} -4 & 0 & 4 \\ 1 & 4 & 2 \\ -1 & 5 & -3 \end{pmatrix} \end{array}$$

Notes

Solution: Find the smallest element in rows and largest elements in columns as shown in Table below.

n Minimax Procedure

		Player B			
		B_1	B_2	B_3	Row Min
Player A	A_1	– 4	0	4	– 4
	A_2	1	4	2	1
	A_3	–1	5	–3	–3
		1	5	4	

Select the largest element in row and smallest element in column. Check for the minimax criterion,

$$\text{Max Min} = \text{Min Max}$$

$$1 = 1$$

Therefore, there is a saddle point and it is a pure strategy.

Optimum Strategy:

Player A $\rightarrow$ A_2 Strategy

Player B $\rightarrow$ B_1 Strategy

The value of the game is 1.

Did u know?

1. The value of the game, in general satisfies the equation, maximin value<=V<= minimax value.
2. A Game is said to be fair game if the lower (maximin) and upper(minimax) values of the game are equal and both equal zero.
3. A game is said to be strictly determinable if the lower (maximin) and upper (minimax) values of the game are equal and both equals the value of the game.

Task Check whether the following game is given in below Table , determinable and fair.

		Player B		
		1	2	Row Min
Player A	1	7	0	0
	2	0	8	0
Column Max		7	8	

Notes

9.4 Two-Person-Zero-Sum Games of 2 x m and n x 2

A game in which one of the two players has 2 strategies whilst opponent has more than 2 (say m) is called 2 × m or n × 2 game depending upon whether columns or rows offer more choices.

For example, The following payoff matrices are of 2 × 3 and 3 × 2 size

$$A = \begin{pmatrix} 1 & 3 & 11 \\ 8 & 5 & 2 \end{pmatrix} \qquad \text{and } A \begin{pmatrix} 1 & 4 \\ 3 & 2 \\ 3 & 4 \end{pmatrix}$$

In such cases we have to find the saddle point. If it exists give the optimal strategies of the players. If no saddle point exists for (2 x m) or (x x 2) game then we have to reduce the game to (2 x 2) which can be done by using (1) Dominance method, (2) Sub-game method and (3) Graphic Method.

Dominance Method

Dominance method specifies the rules to be followed to eliminate strategies without affecting the final solution.

Situations arise when some strategies dominate others in the payoff matrix. A strategy is said to dominate another strategy if all its possible outcomes are preferable. In case of player A, row is said to be dominating row if every element of that row is higher than or equal to the corresponding entries of the another row called the dominating row. A column is said to be dominating if every element of that column is lower than or equal to corresponding elements of another column called the dominating column. Since the dominated strategies will never be chosen, they can be removed from the payoff matrix to simplify the game. Here, we have to proceed row by row and column by column and identify dominated rows and columns and delete them and reduce the size of the payoff matrix. If it is reduced 2×2 matrix, use algebraic method or minimax method to find the optimal solution. If not, use sub-game method or graphical method to reduce the matrix further:

Example:

$$\begin{array}{c} \text{B} \\ A = \begin{pmatrix} 3 & 5 & 4 & 2 \\ 5 & 6 & 2 & 4 \\ 2 & 1 & 4 & 0 \\ 3 & 3 & 5 & 2 \end{pmatrix} \end{array}$$

Step 1: All the entries of 1 row are higher than that of III row. Hence, III row is dominated by 1 row. Hence eliminate III row.

$$\begin{array}{c} \text{B} \\ A = \begin{pmatrix} 3 & 5 & 4 & 2 \\ 5 & 6 & 2 & 4 \\ 3 & 3 & 5 & 2 \end{pmatrix} \end{array}$$

Step 2: All the elements of I column are less than that of II column. Hence, eliminate II column.

$$\begin{array}{c} \text{B} \\ A = \begin{pmatrix} 3 & 4 & 2 \\ 5 & 2 & 4 \\ 3 & 5 & 2 \end{pmatrix} \end{array}$$

Notes

Step 3: All the entries of III row are higher or equal to I row. Hence eliminate I row.

$$A = \begin{pmatrix} 5 & 2 & 4 \\ 3 & 5 & 2 \end{pmatrix}$$

Step 4: All the entries of III column are lower than that of I column. Hence, eliminate I column.

	B	Row Min.
A	2 4	2
	5 2	2
Col. Max.	5 4	

Minimax = 4

Maximin = 2

Therefore Minimax is not equal to Maximin. Hence, no saddle point exists. Hence, we use algebraic method to find the value of the game.

Let x and (1 – x) and y and (1 – y) denote the probabilities of selecting the first and second strategies by the players A and B respectively.

$$x = \frac{P_{22} - P_{21}}{(P_{22} - P_{21}) + (P_{11} - P_{12})}$$

$$= \frac{(2-5)}{(2-5)+(2-4)}$$

$$= \frac{-3}{(-3)+(-2)}$$

$$= \frac{-3}{-5} = \frac{3}{5}$$

Therefore $x = \frac{3}{5}$ and $(1 - x) = \frac{2}{5}$

$$y = \frac{P_{22} - P_{12}}{(P_{22} - P_{12}) + (P_{11} - P_{12})}$$

$$= \frac{(2-4)}{(2-4)+(2-5)}$$

$$= \frac{-2}{-2+-3}$$

$$= \frac{-2}{-5} = \frac{2}{5}$$

$$y = \frac{2}{5} \text{ and } (1 - y) = \frac{3}{5}$$

$$\text{Value of the game} = \frac{(P_{22}P_{11} - P_{12}P_{21})}{(P_{22} - P_{21}) + (P_{11} - P_{12})}$$

$$= \frac{2(2) - 4(5)}{(2-5)+(2-4)}$$

Notes

$$= \frac{4-20}{-3+-2}$$

$$= \frac{-16}{-5} = \frac{16}{5}$$

$$V = 3.2$$

Caution Rules of dominance discussed are used when the payoff matrix is a profit matrix for the player A and a loss matrix for player B. Otherwise the principle gets reversed.

Self Assessment

Multiple Choice Questions:

9. What happens when maximin and minimax values of the game are same?
   (a) No solution exists (b) Solution is mixed
   (c) Saddle point exists (d) None of the above
10. In a mixed strategy game
   (a) No saddle point exists
   (b) Each player selects the same strategy without considering other player's choice
   (c) Each player always selects same strategy
   (d) All of the above
11. When the sum of the gains of one player is equal to the sum of losses to another player in the game, this situation is known as
   (a) Biased game (b) Zero-sum game
   (c) Fair game (d) All of the above

9.5 Limitations of Games in Competition

1. In practice, the two competitors rarely have equal information or intelligence.
2. There may be situations in the managerial decisions when both the players may lose or gain. Hence, it may not be zero-sum game.
3. Most of the competitive situations in which managerial decisions are taken are never really two-person games as government or other bodies are always present as third or fourth party in the game.
4. The technique of solving games involving mixed strategies is very complicated specially for problems having large payoff matrices.
5. The principle of minimax implies that the businessman and his chances of maximum loss but a dynamic businessman is always after profit and tries to maximize his gains or profit even by taking greater risks.

In spite of these limitations the technique has wider applications as it helps us to learn how to approach and understand a conflict situation and how to improve our decision process.

Notes

9.6 Summary

- Game theory is the study of the ways in which strategic interactions among rational players produce outcomes with respect to the preferences of those players, none of which might have been intended by any of them.
- Game theory describes the situations involving conflict in which the payoff is affected by the actions and counteractions of intelligent opponents.
- Two-person zero-sum games play a central role in the development of the theory of games.
- There are many ways to solve a game theory problem, for e.g., that of linear programming, matrix or graphical.
- Every way has its own significance and advantages, thus making game theory a model for winning business in a competitive environment.

9.7 Keywords

Minimax Criterion: Minimax criterion is selecting the strategies that minimize the loss for each player. In other words, the player always anticipates worst possible outcome and chooses the strategy to get maximum for profit and minimum for loss.

Mixed Strategy: Selects at least 2 courses of action, while a probability of selecting an individual strategy will be less than 1, but the sum of the strategies will be 1.

Pure Strategy: The strategy to select a course of action with the probability of one.

Saddle Point: A situation where both the players are facing pure strategies.

Strategy: All possible actions that can be taken for every pay–off.

***Value of the game*:** The Value of the game is the expected gain of player A if both players use their best strategies. The best strategy is arrived at using minimax criterion.

9.8 Review Questions

1. What is a game in game theory? What are the properties of a game? Explain the "best strategy" on the basis of minimax criterion of optimality.
2. What are pure and mixed strategies and how do they differ?
3. When do you say that the game is fair? Give an example for it.
4. Explain the following:
   (a) Saddle point
   (b) Pay-off matrix
   (c) Rectangular games
   (d) Optimal strategy
   (e) Minimax principle
   (f) Value of a two-person-zero-sum game

Notes

5. How do you solve a game when:
   (a) a saddle point exists, and
   (b) Saddle point does not exist.
6. Explain the theory of dominance in the solution of Rectangular games.
7. What are the limitations of Game theory?
8. Consider the game G with following payoff matrix:

Player B

Player A		B_1	B_2
	A_1	2	6
	A_2	–2	μ

   (a) Show that G is strictly determinable whatever μ be.
   (b) Determine the value of G.
9. Determine the optimal strategies for each player in the following game:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{pmatrix} -5 & 2 & 0 & 7 \\ 5 & 6 & 4 & 8 \\ 4 & 0 & 2 & -3 \end{pmatrix} \end{array}$$

   Ans. [Saddle Pt. is (2, 3) and V = 4 optimum strategy for player A is A_2 and that for B is B_3
10. Solve the following games:

(a)

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\ \text{Player A} \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \\ \text{V} \end{array} & \begin{pmatrix} 4 & 0 & 1 & 7 & -1 \\ 0 & -3 & -5 & -7 & 5 \\ 3 & 2 & 3 & 4 & 3 \\ -6 & 1 & -1 & 0 & 5 \\ 0 & 0 & 6 & 0 & 0 \end{pmatrix} \end{array}$$

(b)

$$\begin{array}{cc} & \text{B} \\ \text{A} & \begin{pmatrix} 2 & -1 & 3 \\ 2 & -1 & 2 \\ -1 & 0 & 0 \end{pmatrix} \\ & \begin{array}{ccc} 2 & 0 & 4 \end{array} \end{array}$$

(c)

$$\text{A} \begin{pmatrix} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -4 & -2 & 0 & -5 \end{pmatrix}$$

11. Find the saddle point for the following payoff table:

	1	2	3	4
1	2	–1	0	–3
2	2	1	0	3
3	–3	–2	1	4

Notes

12. Solve the game whose payoff matrix is given by,

$$\begin{array}{cc} & \text{B} \\ & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{array}{c} \text{I} \\ \text{A II} \\ \text{III} \end{array} & \begin{pmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{pmatrix} \end{array}$$

13. Solve the following payoff table and obtain the value of the game where $0 < a_1 < a_2 < a_3 < \ldots < a_5 < a_6$

$$\begin{pmatrix} a_4 & a_3 & a_3 \\ a_1 & a_6 & a_5 \\ a_2 & a_4 & a_3 \end{pmatrix}$$

14. Solve the game:

$$\begin{array}{cc} & \text{B} \\ \text{A} & \begin{pmatrix} 9 & 1 & 4 \\ 0 & 6 & 3 \\ 5 & 2 & 8 \end{pmatrix} \end{array}$$

15. Solve the following game:

$$\begin{array}{cc} & \text{B} \\ \text{A} & \begin{pmatrix} 6 & 2 & 7 \\ 3 & 4 & 8 \\ 5 & 4 & 1 \end{pmatrix} \end{array}$$

16. There are two competing stores R and C in a city. Both the stores have equal reputation and the total number of customers is equally divided between the two. Both the stores plan to run annual discount sale on the last week of December. For this they want to attract more number of customers by using advertisement through newspaper, radio and television. By seeing the market trend, the store R constructed the following payoff matrix where the numbers in the matrix indicate a gain or loss of customers.

$$\begin{array}{cc} & \text{Strategies of Store 'C'} \\ \text{Strategies of Store 'R'} & \begin{pmatrix} & C_1 & C_2 & C_3 \\ R_1 & 40 & 50 & -70 \\ R_2 & 10 & 25 & -10 \\ R_3 & 100 & 30 & 60 \end{pmatrix} \end{array}$$

Find the optimal strategies for both the stores along with the value of the game.

Answers: Self Assessment

1.	Strategy	2.	Pure Strategy
3.	Mixed strategy	4.	Saddle Point
5.	Value of the Game	6.	False
7.	True	8.	False
9.	(c)	10.	(a)
11.	(b)		

9.9 Further Readings

Notes

Books

J.K. Sharma, *Operations Research, Theory and Applications,* MacMillan India Ltd.

Kanti Swarup, P.K Gupta & Manmohan, *Operations Research*, Sultan Chand Publications, New Delhi

Michael W. Carter, Camille C. Price, *Operations Research: A Practical Introduction,* CRC Press, 2001

Paul A. Jensen, Jonathan F. Bard, *Operations Research Models and Methods,* John Wiley and Sons, 2003

Richard Bronson, Govindasami Naadimuthu, *Schaum's Outline of Theory and Problems of Operations Research,* McGraw-Hill Professional; 1997

Online links

http://www.gametheory.net/lectures/level.pl

http://plato.stanford.edu/entries/game-theory/

www.game-theory-class.org/

Notes

Unit 10: Sequencing Problems and Replacement Theory

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Objectives

After studying this unit, you will be able to:

- Appreciate the aims to study sequencing problems
- Use Johnson's rule of sequencing
- Solve some specific problems of scheduling jobs on one machine, two and three machines
- Learn about the application of replacement theory.

Introduction

Sequencing deals with the selection of an optimum order for the number of jobs to be performed with a finite number of facilities. The objective of sequencing is to determine the sequence of performing jobs such that the total cost/time is minimized.

Sequencing problems arise in situation like a manufacturing plant for performing orders, in a hospital where the patients wait for different type of tests and in a bank where customers wait for different types of transactions, etc.

10.1 Terminology

- ***Processing Time*** t_{ij} : It is the time required by a job i on each machine j.

Notes

- ***Total Elapsed Time T:*** It is the time interval between starting the first job and completing the last job including the idle time (if any) in a particular order by the given set of machines.
- ***Idle Time I_{ij}:*** It is the time for which a machine *j* does not have a job to process, i.e., idle time from the end of job *(i-1)* to the start of job *i.*
- ***Processing Order:*** It refers to the order or sequence in which machines are required for completing the job.
- ***Number of Machines:*** It refers to the service facilities available through which a job must pass before it is assumed to be completed.
- ***No Passing Rule:*** It refers to the condition to be satisfied in which the order of jobs are to be processed on given machines.

Assumptions

There are some assumptions made while solving sequencing problems.

1. The processing times on different machines are exactly known and are independent of the order of the jobs in which they are to be processed.
2. The time taken by the jobs in moving from one machine to another is negligible.
3. Once a job has begun in a machine, it must be completed before another job can begin on the same machine.
4. A job starts on a machine as soon as the previous job is over, and the machine is free to work.
5. No machine can process more than one job simultaneously.
6. Machines to be used are of different types.
7. The order of completion of jobs are independent, i.e., no job is given priority.

As discussed earlier, the objective of sequencing is to determine the sequence of jobs so that all the jobs are performed in a minimum completion time from the beginning of the first job till the completion of the last job.

The advantages of sequencing are:

1. We get the information like which machines are idle and period of idle time.
2. Completion time needed for all the jobs to be completed.

Notes There are also some other objectives of scheduling (or sequencing) such as minimizing customers wait time, meeting promised delivery dates, keeping stock level low, giving preferred working pattern, minimizing waiting time of patients in a hospital for different types of tests and so on.

Task Give three different examples of sequencing problems from your daily life.

Notes

Self Assessment

Give one word answer for the following:

1. The time interval between starting the first job and completing the last job including the idle time (if any) in a particular order by the given set of machines.
2. Order or sequence in which machines are required for completing the job.
3. The condition to be satisfied in which the order of jobs are to be processed on given machines.

10.2 Types of Sequencing Problems

Type I: Processing *n* jobs through *two* machines

Type II: Processing *n* jobs through *three* machines

Type III: Processing *n* jobs through *m* machines

Type IV: Processing *two* jobs through *m* machines

Notes In this unit we will only discuss the type I and II by Johnson's Rule. Type III and IV are complex problems of sequencing and are also solved using extension of Johnson's Rule.

10.3 Algorithm for Sequencing Problems

10.3.1 Type I: Processing n Jobs through Two Machines

Considering *n* jobs (*say* 1,2, …n) to be processed on *two* machines A and B, in the order AB. The processing time *is* $t_{11}, t_{12,\ldots} t_{1n}$ *and* $t_{12}, t_{22} \ldots\ldots t_{2n}$ as represented in Table 10.1 below:

Table 10.1: Sequencing Problem

Machines	Job				
	1	2	3	….	n
A	t_{11}	t_{12}	T_{13}	….	t_{1n}
B	t_{21}	t_{22}	T_{23}	….	t_{2n}

Step 1: Check the processing order, i.e., in the order AB or BA. If it is in the order AB, then the first job would be performed on machine A and then on machine B. If the order is of BA, interchange the rows with machine B in row l and machine A in row 2.

Step 2: From the given processing time, select the least processing time available on both machines A and B. If the least processing time exists in row l, place that job at the beginning of the sequence table (i.e., the sequence is from left to right of the sequence table). If the least processing time is on row 2, place the job at the end of sequence table (i.e., the sequence is from right to left of the sequence table).

Note: If there are two least processing times for machine A and machine B, priority is given for the processing time which has lowest time of the adjacent machine.

Notes

Step 3: Delete the job which has been sequenced, and repeat step 2 until all the jobs are sequenced.

***Step 4*:** Establish a tabular column to determine the total elapsed time and also the idle time for both the machines A and B.

1. ***Total Elapsed Time:*** Time when the last job in the sequence has finished on Machine B.
2. ***Idle Time for Machine A:*** (Total Elapsed Time) – (Time when the last job has finished on machine B)
3. ***Idle Time for Machine B:*** Time at which the first job is finished on machine $A + \sum_{j=2}^{n}(\text{Time when the job starts on machine B})$ – (Time when the (j – 1)th job finishes on machine B)

Example: There are five jobs, each of which must go through the two machines A and B in the order AB. Processing times (in hours) are given in the table 10.2 below:

Table 10.2: Sequence Problem

Job	1	2	3	4	5
Machine A	5	1	9	3	10
Machine B	2	6	7	8	4

Determine a sequence for the five jobs that will minimize the elapsed time.

Solution:

In the given problem, the jobs must be processed through the machines A and B in the order AB, i.e., the first machine to be processed is A and then B. Establish a sequence table containing 5 job cells to sequence the jobs. Now, select the least processing time considering both the machines. The least time is taken by job 2 on machine A (i.e., 1). Place job 2 in the first cell from left to right of the Table 10.3 (since it occurs in row 1).

Table 10.3: Job 2 has Least Processing Time

2				

Machine A → ← Machine B

Delete the allotted Job 2. Now select the least processing time available in the existing Table 10.4.

Table 10.4: Reduced Table Column for Job 2 Deleted

Job	**1**	**3**	**4**	**5**
Machine A	5	9	3	10
Machine B	2	7	8	4

The least processing time in the reduced Table 10.5 is 2 for Job l, on machine B. Place this job 1 from right to left in the last cell (since it occurs in row 2).

Notes

Table 10.5: Job 1 has Least Processing Time

2				1

The new reduced table after deleting job 1 is given (Table 10.6)

Table 10.6: Reduced Table Column for Job 1 Deleted

Job	3	4	5
Machine A	9	3	10
Machine B	7	8	4

The least time is 3 for Job 4, on machine A. Place this job left to right (since it occurs in row 1) next to job 2 as shown in Table 10.7.

Table 10.7: Job 4 has Least Processing Time

2	**4**			**1**

The reduced table after deleting job 4 is given in Table 10.8.

Table 10.8: Reduced Table, Job 4 Deleted

Job	3	5
Machine A	9	10
Machine B	7	4

The least processing time is 4 for Job 5. Place the job right to left (since it occurs in row 2) next to job 1 as shown in Table 10.9.

Table 10.9: Job 5 has Least Processing Time

2	4		5	1

Finally we have only one job left, that is Job 3. Place the job in the empty cell as shown in Table 10.10.

Table 10.10: Job 3 Sequenced Optimal Sequence

2	4	3	5	1

The optimal sequence is thus obtained and the flow of Job 1 to Job 5 through the machine A and the machine B will be

$$2 \rightarrow 4 \rightarrow 3 \rightarrow 5 \rightarrow 1$$

The minimum elapsed time and the idle time for machine A and machine B is shown in Table 10.11.

Notes

Table 10.11: Final Optimal Sequence Table

Job Sequence	Machine A		Machine B		Idle Time A	Idle Time B
	Time In	Time Out	Time In	Time Out		
2	0	1	1	7	0	1
4	1	4	7	15	0	0
3	4	13	15	22	0	0
5	13	23	23	27	0	1
1	23	28	28	30	0	1
					30-28=2	
					2	**3**

In Table 10.11, the minimum elapsed time, (from Job 2 to Job 1) is 30 hours. The idle time on machine A is 2 hours and on machine B is 3 hours.

Example: Six jobs go first over machine I and then over machine II. The order of the completion of jobs has no significance. The table 10.12 shows the machine times in hours for six jobs and the two machines.

Table 10.12: Sequence Problem

Job	J_1	J_2	J_3	J_4	J_5	J_5
Machine I	2	4	9	6	7	4
Machine II	6	7	4	3	3	11

Find the sequence of the jobs that minimizes the total elapsed time to complete the jobs. Also find the idle time for Machine I and Machine II.

Solution:

Establish a sequence table containing six job cells. Find the least time available for both Machine I and Machine II. Job 1 has the least processing time, i.e., 1. Place the sequence from left to right (or the first cell) as shown in Table 10.13, since it occurs Machine I.

Table 10.13: Job 1 has Least Processing Time

J_1					

Deleting job 1, we get the reduced table as shown in Table 10.14

Table 10.14: Reduced Table, Job 1 Deleted

Job	J_2	J_3	J_4	J_5	J_6
Machine I	3	8	5	6	3
Machine II	6	3	2	2	10

The least time available in the reduced table is 2, which is on Job 4 and Job 5 for Machine II. Now, compare the adjacent time available for Machine I. Here, Job 4 time is less than that for Job 5. Select Job 4 first and sequence it as shown in Table 10.15.

Notes

Table 10.15: Job 4 Sequenced

J_1					J_4

Now select Job 5 and sequence it as shown in Table 6.16.

Table 10.16: Job 5 Sequenced

J_1				J_5	J_4

The reduced table is shown in Table 10.17

Table 10.17: Reduced Table, Job 1, 4 & 5 Deleted

Job	J_2	J_3	J_6
Machine I	3	8	3
Machine II	6	3	10

Now we have Job 2, Job 3 and Job 6 having the least time which is 3. Compare these times with the adjacent machine time and select the least time. Here we have Job 2 with least adjacent time and hence sequence is as shown below in Table 10.18.

Table 10.18: Job 2 Sequenced

J_1	J_2			J_5	J_4

Now select Job 3 and sequence it, as shown in Table 6.19.

Table 10.19: Job 3 Sequenced

J_1	J_2		J_3	J_5	J_4

Finally, select Job 6 and sequence it, as shown in Table 6.20.

Table 10.20: Job 6 Sequenced

J_1	J_2	J_6	J_3	J_5	J_4

The optimal sequence thus obtained is,

$$J_1 \rightarrow J_2 \rightarrow J_6 \rightarrow J_3 \rightarrow J_5 \rightarrow J_4$$

The total elapsed time and idle time for Machine I and Machine II is calculated as shown in Table 10.21 below:

Notes

Table 10.21: Final Optimal Sequence Table

Job Sequence	Machine I		Machine II		Idle Time A	Idle Time B
	Time In	Time Out	Time In	Time Out		
J_1	0	2	2	8	0	2
J_2	2	6	8	15	0	0
J_6	6	10	15	26	0	0
J_3	10	19	26	30	0	0
J_5	19	26	30	33	0	0
J_4	26	32	33	36	0 36-32=4	0

The total elapsed time is **36** hours

Idle time for Machine I is **4** hours

Idle time for Machine II is **2** hours

Self Assessment

State true or false

4. One machine – one job is not an assumption in solving sequencing problems
5. If there are two least processing times for machine A and machine B, priority is given for the processing time which has lowest time of the adjacent machine.
6. There can be four types of sequencing problems depending upon number of jobs and number of machines.

10.3.2 Type II: Processing n Jobs through Three Machines

In this type, we have three machines instead of two with n jobs. All the jobs are to be processed through three machines, say M_1, M_2 and M_3.

Considering n jobs (say 1,2,3….n) to be processed on three machines M_1, M_2 and M_3 in the order M_1, M_2, M_3, the processing time is t_{11}, t_{12},….. t_{1n} and t_{31}, t_{32} ….. t_{3n} as shown in Table 10.22

Table 10.22: Sequence Problem

Machines	Job				
	1	2	3	….	n
M_1	t_{11}	t_{12}	T_{13}	….	t_{1n}
M_2	t_{21}	t_{22}	T_{23}	….	t_{2n}
M_3	t_{31}	t_{32}	T_{33}	….	t_{3n}

Notes

The optimal solution can be obtained if either one of the following conditions is satisfied:

Condition 1:	Minimum time on row 1 should be greater than or equal to maximum time on row 2
	i.e., Min $t_{1j} \geq$ Max t_{2j}, j = 1,2,….n
	(or)
Condition 2:	Minimum time on row 3 should be greater than or equal to maximum time on row 2
	i.e., Min $t_{3j} \geq$ Max t_{2j}, j = 1,2,….n

Algorithm

Step 1: Check whether the given processing time for all the three machines satisfy either one or both the conditions. If the condition is satisfied, go to Step 2, else the model fails.

Step 2: Convert the three machine types into two machines by introducing two imaginary machines M_x and M_y.

Where, $M_x = M_1 + M_2$ and

$M_y = M_2 + M_3$

Step 3: For the imaginary machines M_x and M_y, determine the optimal sequence using the algorithm for Type 1 (i.e., n job and two machines).

Step 4: Find the total elapsed time and the idle time for all the three machines.

Example: A machine operator has to perform three operations — turning, threading and knurling — on a number of different jobs. The time required to perform these operations (in minutes) for each job is given below in Table 10.23. Determine the order in which the jobs should be processed in order to minimize the total time required to perform all the jobs. Also find the minimum elapsed time.

Table 10.23: Sequence Problem

Job	1	2	3	4	5	6
Turning	3	12	5	2	9	11
Threading	8	6.	4	6	3	1
Knurling	13	14	9	12	8	13

Solution:

Initially, check whether the given problem satisfies the condition or not. We have three separate machines for Turning, Threading, and Knurling, so let these three machines be M_1, M_2 and M_3 respectively.

Check the condition for,

Minimum time for $M_1 \geq$ Maximum time M_2

i.e., $2 \geq 8$ does not satisfy the condition.

(or)

Minimum time for $M_3 \geq$ Maximum time on M_2

i.e., $8 \geq 8$ satisfies the condition.

Notes

Since either one of the conditions is satisfied, the problem can now be converted into two-machine type.

Assuming imaginary machines as M_4 and M_5 (see Table 6.33,)

Where, $M_4 = M_1 + M_2$ and,

$M_5 = M_2 + M_3$

Table 10.24: Problem Converted to 2-Machine Type

Job	1	2	3	4	5	6
M_4	3+8=11	12+6=18	5+4=9	2+6=8	9+3=12	11+1=12
M_5	8+13=21	6+14=20	4+9=13	6+12=18	3+8=11	1+13=14

Now using the algorithm for n jobs and two machines, the optimal sequence is shown in Table 10.25.

Table 10.25: Optimal Sequence

4	3	1	6	2	5

Find the total elapsed time and idle time for machine M_1, M_2 and M_3. These are shown in Table 10.26, below:

Table 10.26: Final Optimal Sequence Table

Job Sequence	Machine A		Machine B		Machine C		Idle Time A	Idle Time B	Idle Time C
	Time In	Time Out	Time In	Time Out	Time In	Time Out			
4	0	2	2	8	8	20	0	2	8
3	2	7	8	12	20	29	0	0	0
1	7	10	12	20	29	42	0	0	0
6	10	21	21	22	42	55	0	1	0
2	21	33	33	39	55	69	0	11	0
5	33	42	42	45	69	77	0	3	0
							77 – 42 = 35	77 – 45 = 32	
							35	**49**	**8**

From the above table, we find that

Total elapsed time = 77 min.

Idle Time for Machine M_1 = 35 min.

Idle Time for Machine M_2 = 49 min.

Idle Time for Machine M_3 = 8 min.

Notes

Task Determine the optimal sequence of jobs that minimize the total elapsed time based on the following information (see Table). Processing time on machines is given in hours, and passing is not allowed.

Sequence Problem

Job	A	B	C	D	E	F	G
Machine M_1	3	8	7	4	9	8	7
Machine M_2	4	3	2	5	1	4	3
Machine M_3	6	7	5	11	5	6	12

10.4 Replacement Theory

Men, materials and machines get deteriorated as the time passes. Hence, they require replacement. Replacement is necessary because they may not be in a position to generate the same level or degree of efficiency which they show throughout the life and before replacement. The exact age or time for replacement is needed to be calculated and sometimes it becomes a critical issue, especially in case of machines and equipments, because the resale value decreases as the decision to replace gets postponed.

Replacement Model helps in determining the optimum time of replacement by considering the running costs and the capital cost of purchasing the equipment. These associated costs can be expressed as the average costs and this average cost goes on decreasing with the postponement of replacement decision. But a time will come when the average cost starts increasing calling the owner to replace the machine or equipment.

10.4.1 Need for Replacement

1. The existing materials or machines might have out lived their effective lives and it may not be economically feasible to allow them to continue in the organization.
2. The existing materials or equipments might have been destroyed by accident.

$$\text{Total Cost p.a.} = \text{Maintenance cost} + \text{Loss in purchase price}$$

$$\text{Average cost} = \frac{\text{Total Cost}}{\text{No. of years of equipment life}}$$

$$\text{Running cost} = R_n$$

$$SR_n = \text{Total running cost [cumulative Rn]}$$

$$C - S = \text{Capital cost of acquisition} - \text{Scrap/resale price}$$

$$P_{(n)} = \text{Total cost}$$

$$\frac{P(n)}{n} = \frac{\text{Total Cost}}{\text{No. of years}} \Big\} \text{Average cost}$$

Justification

An inference is to be arrived at to replace the machine or equipment when the cost of maintaining a particular machine during a particular year is more than the average cost of equipment for the year concerned.

Notes

1st Case: Theorem: Statement

The cost of maintenance of a machine is given as a function increasing with time and its scrap value is constant.

1. If the time is measured continuously, then the average annual cost will be minimized by replacing the machine when the average cost to that date becomes equal to the current maintenance cost.
2. If the time is measured in discrete units, then the average annual cost will be minimized by replacing the machine when the next period's maintenance cost becomes greater than the current average cost.

Example: The cost of a machine is ₹ 6, 100 and its scrap value is ₹ 100 only. The maintenance costs are found from experience to be:

Year	1	2	3	4	5	6	7	8
Maintenance Cost (₹)	100	250	400	600	900	1,250	1,600	2,000

Solution:

Table showing optimum period of replacement of the machine

Replacement at the end of the year	Running cost R_n (₹)	Total R_n ΣR_n (₹)	C – S (₹)	Total cost P_n (₹)	Average cost (₹)
1	100	100	6,000	6,100	6,100
2	250	350	6,000	6,350	3,175
3	400	750	6,000	6,750	2,250
4	600	1,350	6,000	7,350	1,837.05
5	900	2,250	6,000	8,250	1,650
6	1,250	3,500	6,000	9,500	1,583.33
7	1,600	5,100	6,000	11,100	1,585.07
8	2,000	7,100	6,000	13,100	1,637.05

Inference

The machine has to be replaced at the end of the 6th year or at the beginning of the 7th year as the maintenance cost of the 7th year becomes higher than the average cost of maintaining the machine at the end of the 6th year.

Example: A machine owner finds from his past records that the costs per year of maintaining a machine different and which are given below:

Year	1	2	3	4	5	6	7	8
Maintenance cost in (₹)	1,000	1,200	1,400	1,800	2,300	2,800	3,400	4,000
Resale price (₹)	3,000	1,500	750	375	200	200	200	200

If the purchase price of that machine is ₹ 6,000, then at what age the replacement is due?

Notes

Solution:

n	R_n	ΣR_n	C – S	P_n	Average cost
1	1,000	1,000	3,000	4,000	4,000
2	1,200	2,200	4,500	6,700	3,350
3	1,400	3,600	5,250	8,850	2,950
4	1,800	5,400	5,625	11,025	2,756.25
5	2,300	7,700	5,800	13,500	2,700
6	2,800	10,500	5,800	16,300	2,716.67
7	3,400	13,900	5,800	19,700	2,814.28
8	4,000	17,900	5,800	23,700	2,962.05

Inference

The machine has to be replaced at the end of the 5th year or at the beginning of the 6th year as the maintenance cost of the 6th year becomes higher than the average cost of maintaining the machine at the end of the 5th year.

Self Assessment

7. A firm is considering replacement of a machine whose cost is ₹ 12, 200 and scrap value is ₹ 200. The maintenance costs are found from experience to be as follows:

Year	1	2	3	4	5	6	7	8
Maintenance cost (₹)	200	500	800	1,200	1,800	2,500	3,200	4,000

When should the machine be replaced?

a. At the end of 6th year or beginning of 7th year

b. At the end of 4th year

c. At the end of 8th year

d. At the beginning of 5th year

Task The following table gives the running costs per year and resale price of a certain equipment. Its purchase price is ₹ 5,000

Year	1	2	3	4	5	6	7	8
Running costs (Rs.)	1,500	1,600	1,800	2,100	2,500	2,900	3,400	4,000
Resale value (Rs.)	3,500	2,500	1,700	1,200	800	500	500	500

At what year the replacement is due?

Notes

10.5 Consideration of Money Value for Investments on Machines or Equipments

10.5.1 Replacement Policy Decisions

Replacement policy for items whose maintenance cost increases with time and the money value changes with constant rate. If the maintenance cost increases with time and the money value decreases with constant rate, the replacement policy will be:

1. Replace if the next period's cost is greater than the weighted average cost of the previous period.
2. Do not replace if next period's cost is less than the weighted average cost of the previous period.

How to select the Best Machine?

1. Weighted average cost of each machine is to be computed.
2. If the weighted average cost of X_1 [1st machine] is less than X_2 [2nd machine] then choose X_1.
3. If the weighted average cost of X_1 is greater than X_2 than choose machine B.
4. If the weighted average cost of X_1 and X_2 are equal then both the machines are good.

Notations used:

n = Year of replacement

R_n = Running cost

V^{n-1} = Present worth factor(PWF)

$R_n V^{n-1}$ = The running costs at discounted rates

$\Sigma R_n V^{n-1}$ = Cumulative running costs at discounted rates

TC = Total Cost – Capital Cost + $\Sigma R_n V^{n-1}$

WAC = Weighted average cost

$$= \frac{TC}{\Sigma V^{n-1}}$$

Example: The following table has the details of 2 machines A and B:

Year	Cost at the beginning of the year	
	Machine A	Machine B
1	900	1,400
2	600	100
3	700	700

Find the cost pattern for each machine when money is worth 10% per year and find which machine is less costly.

Solution:

1. If the future value of money is not considered, then the decision as to which machine is best is arrived at by comparing the total outlay of both machines which is worked out as under:

Notes

Year	Total outlay	
	Machine A	Machine B
1	900	1,400
2	600	100
3	700	700
Total outlay	2,200	2,200

As the total outlay of both the machines are equal, both the machines are good.

2. When future value of money is considered at a constant rate of 10% p.a., then the total outlay differs. The following calculations support of this view:

Year	Total outlay		Working
	Machine A	Machine B	
1	900.00	1,400.00	$600 \times \frac{100}{100} = 545.00$
2	545.45	90.91	$100 \times \frac{100}{100} = 90.91.00$
3	578.51	578.51	$700 \times \left(\frac{100}{110}\right)^2 = 578.51$
Total	2,023.96	2,069.42	

Inference

It is evident from the above table that the outlay on machine A is better than machine B. Hence, machine A is better than machine B.

Example: A manufacturer is offered 2 machines A and B. A is priced at ₹ 5,000 and the running costs are estimated at ₹ 800 for each of the 1st year, increasing by ₹ 200 per year in the 6th and subsequent years. Machine B, has the same capacity as A, costs ₹ 2,500. But, it will have running costs of ₹ 1,200 per year for 6 years increasing by ₹ 200 per year thereafter.

If money is worth 10% per year, which machine should be purchased [assume that machines will eventually be sold for scrap at a negligible price]?

Solution:

Calculation of weighted average cost for machine A

N	R_n	V^{n-1}	R_nV^{n-1}	ΣR_nV^{n-1}	$C+\Sigma RV^{n-1}$	ΣV^{n-1}	$TC/\Sigma V^{n-1}$
1	800	1.000	800	800	5,800	1.0000	5,800
2	800	0.9091	727.28	1,527.28	6,527.28	1.9091	3,419.04
3	800	0.8264	661.12	2,188	7,188.4	2.7335	2,627.8
4	800	0.7513	601.04	2,789.4	7,789.4	3.4868	2,233.98
5	800	0.6830	546.4	3,335.8	8,335.8	4.1698	1,999.09
6	1,000	0.6209	620.9	3,956.7	8,956.7	4.7907	1,869
7	1,200	0.5645	677.4	4,634.1	9,634.14	5.3552	1,799
8	1,400	0.5132	718.48	5,352.6	10,352.6	5.8684	1,762
9	1,600	0.4665	746.4	6,099.02	11,099.02	6.3349	1,752
10	1,800	0.4241	763.38	6,862.41	1,862.4	6.7590	1,503

Notes

Calculation of weighted average cost of machine B

N	R_n	V^{n-1}	R_nV^{n-1}	ΣR_nV^{n-1}	$C+\Sigma RV^{n-1}$	ΣV^{n-1}	$TC/\Sigma V^{n-1}$
1	1,200	1.0000	1,200	1,200	3,700	1.0000	3,700
2	1,200	0.9091	1,090.9	2,290	4,790.9	1.9091	2,509.5
3	1,200	0.8264	991.68	3,282.6	5,782.6	2.7355	2,113.91
4	1,200	0.7513	901.56	4,184.16	6,684.16	3.4868	1,799.55
5	1,200	0.6830	819.6	5,033.76	7,503.76	4.1698	1,916.99
6	1,200	0.6209	745.08	5,748.8	8,248.8	4.7907	1,721.84
7	1,400	0.5645	790.3	6,539.14	9,039.14	5.3552	1,687.92
8	1,600	0.5132	821.12	7,360.26	9,860.3	5.8682	1,680.29
9	1,800	0.4665	839.7	8,199.96	10,699.96	6.3349	1,689.05
10	2,000	0.4241	848.2	9,048.16	11,548.16	6.7590	1,708.56

Inference

It becomes evident from the table that the weighted average cost of machine B is lower than the weighted average cost of machine A. Hence, it can be arrived at a decision that machine B is better than machine A.

Caution If the weighted average cost of X_1 and X_2 are equal then both the machines are equally good.

10.6 Summary

- Sequencing problems can be considered with situations where there is a selection of an appropriate order for a series of different jobs to be done on a finite number of service facilities.
- A sequencing problem involves jobs in a manufacturing unit, aircraft waiting for landing and clearance, maintenance scheduling in a manufacturing company, customers in an electricity department, and so on.
- Consider the sequencing problems in respect of the jobs to be performed in an organization and study the different method of their feasible solution.
- This type of sequencing problems may be divided in two strata. In the first one, there are n jobs to be completed; each job requires processing on some or all of the m different machines.
- Then find out the effectiveness of each of the sequences that are technically correct & feasible and select a sequence which optimizes the total effectiveness of the system.
- To explain, the timings of processing of the n jobs on each of the m machines, in a particular defined order, may be given and the time for completing the jobs may be the measure of effectiveness.
- Arrange the sequence(s) for which the total time consumed by the machines in processing all the jobs.

Notes

- Sequencing problems can be considered with situations where there is a selection of an appropriate order for a series of different jobs to be done on a finite number of service facilities.
- Sequencing helps to resolve problems when multiple jobs are to be done on finite facilities.

10.7 Keywords

Idle Time: Time for which the machine does not have any job to process.

No passing Rule: The condition to be satisfied in which the order of jobs are to be processed on given machines.

Processing Order: Sequence in which the machines are required for completing the job.

Processing Time: Time required by a job on each machine.

10.8 Review Questions

1. What is a sequencing problem?
2. Define the various terminologies used in sequencing.
3. What are the assumptions made in solving sequencing problems ?
4. Explain the method of processing 'n' jobs through two machines.
5. What is the 'no passing rule' in sequencing?
6. Define the terms (i) Total elapsed time, and (ii) Idle time.
7. What are the advantages of sequencing?
8. Find the optimal sequence that minimizes the total elapsed time required to complete the following jobs.

Job	1	2	3	4	5	6
Machine 1	4	6	6	10	5	8
Machine 2	6	7	9	2	4	11

9. A plastic folder manufacturer has two machines – cutting machine and sealing machine. He produces five varieties of folders. Due to design and size, the time taken for cutting and sealing differs from one folder to another. The time consumed for each operation is given in the table. The manufacturer wants to determine the order in which the folders should be processed to minimize the total time required to process all the folders.

Folder Type	1	2	3	4	5
Cutting	15	30	25	10	15
Sealing	10	9	7	3	10

10. An insurance company receives three types of policy application bundles daily from its head office for data entry and filing. The average time taken for each type for these two operations is given in the following table:

Policy	1	2	3
Data Entry	90	120	80
Filing	140	110	100

Notes

The manager of the insurance company is interested in minimizing the total time taken to complete the process.

11. Five jobs have to be processed on two machines A and B in the order AB. Processing times (in minutes) are given in the table below:

Job	1	2	3	4	5
Machine A	6	10	5	8	9
Machine B	8	5	9	4	10

12. What is a replacement problem?
13. Describe various types of replacement problems.
14. What are the situations which make the replacement of item necessary?
15. Discuss the replacement problem where items are such that maintenance costs increase with time and the value of money also changes with time.

Answers: Self Assessment

1. Total elapsed time
2. Processing Order
3. No passing Rule
4. False
5. True
6. False
7. (a) The machine is to be replaced at the end of the 6th year or at the beginning of the 7th year as the maintenance cost of the 7th year becomes higher than the average cost of maintaining the machine at the end of the 6th year.

10.9 Further Readings

Books

J.K. Sharma, *Operations Research, Theory and Applications,* MacMillan India Ltd.

Kanti Swarup, P.K Gupta & Manmohan, *Operations Research*, Sultan Chand Publications, New Delhi

Michael W. Carter, Camille C. Price, *Operations Research: A Practical Introduction,* CRC Press, 2001

Paul A. Jensen, Jonathan F. Bard, *Operations Research Models and Methods,* John Wiley and Sons, 2003

Richard Bronson, Govindasami Naadimuthu, *Schaum's Outline of Theory and Problems of Operations Research,* McGraw-Hill Professional; 1997

Notes

Online links

http://www.universalteacherpublications.com/univ/ebooks/

http://www.jstor.org/pss/2628801

Notes

Unit 11: Queuing Theory

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Objectives

After studying this unit, you will be able to:

- Identify and examine situations that generate queuing problems
- Describe the trade-off between cost of service and cost of waiting line
- Analyze a variety of performance measures of a queuing system
- Understand the concept of single server Queuing model

Introduction

Queuing theory deals with problems that involve waiting (or queuing). It is quite common that instances of queue occurs everyday in our daily life. Examples of queues or long waiting lines might be

- Waiting for service in banks and at reservation counters.
- Waiting for a train or a bus.
- Waiting for checking out at the Supermarket.
- Waiting at the telephone booth or a barber's saloon.

Whenever a customer arrives at a service facility, some of them usually have to wait before they receive the desired service. This forms a queue or waiting line and customers feel discomfort either mentally or physically because of long waiting queue.

Notes

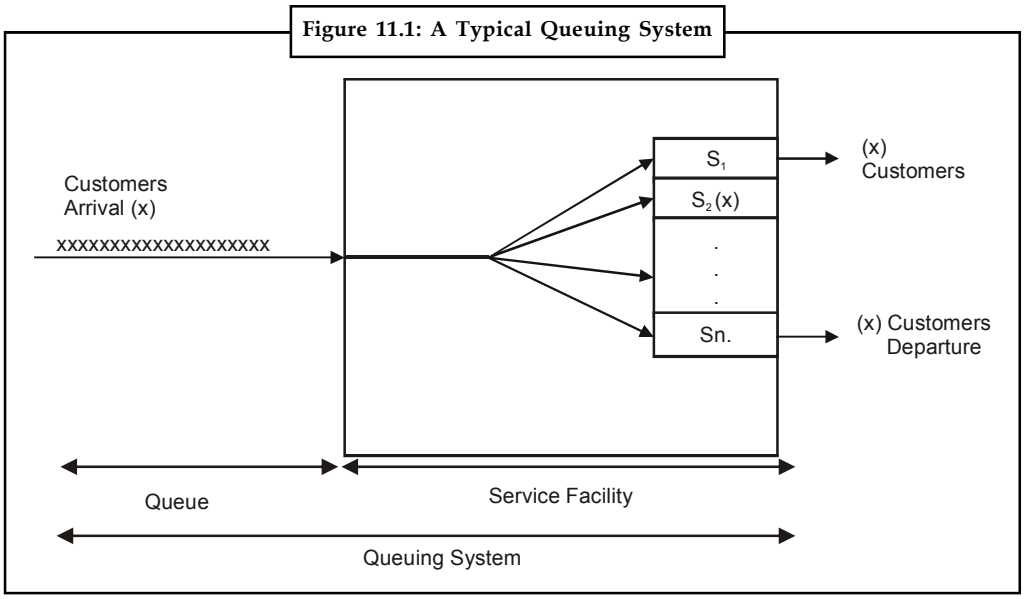
We infer that queues form because the service facilities are inadequate. If service facilities are increased, then the question arises how much to increase? For example, how many buses would be needed to avoid queues? How many reservation counters would be needed to reduce the queue? Increase in number of buses and reservation counters requires additional resource. At the same time, costs due to customer dissatisfaction must also be considered.

In designing a queuing system, the system should balance service to customers (short queue) and also the economic considerations (not too many servers). Queuing theory explores and measures the performance in a queuing situation such as average number of customers waiting in the queue, average waiting time of a customer and average server utilization.

11.1 Queuing System

The customers arrive at service counter (single or in groups) and are attended by one or more servers. A customer served leaves the system after getting the service. In general, a queuing system comprises with two components, the queue and the service facility. The queue is where the customers are waiting to be served. The service facility is customers being served and the individual service stations. A general queuing system with parallel server is shown in Figure 11.1 below:

Figure 11.1: A Typical Queuing System



11.1.1 Characteristics of Queuing System

In designing a good queuing system, it is necessary to have a good information about the model. The characteristics listed below would provide sufficient information.

1. The arrival pattern.
2. The service mechanism.
3. The queue discipline.
4. The number of customers allowed in the system.
5. The number of service channels.

Notes

1. ***The Arrival Pattern:*** The arrival pattern describes how a customer may become a part of the queuing system. The arrival time for any customer is unpredictable. Therefore, the arrival time and the number of customers arriving at any specified time intervals are usually random variables. A Poisson distribution of arrivals correspond to arrivals at random. In Poisson distribution, successive customers arrive after intervals which independently are and exponentially distributed. The Poisson distribution is important, as it is a suitable mathematical model of many practical queuing systems as described by the parameter "the average arrival rate".

2. ***The Service Mechanism:*** The service mechanism is a description of resources required for service. If there are infinite number of servers, then there will be no queue. If the number of servers is finite, then the customers are served according to a specific order. The time taken to serve a particular customer is called the service time. The service time is a statistical variable and can be studied either as the number of services completed in a given period of time or the completion period of a service.

3. ***The Queue Discipline:*** The most common queue discipline is the **"First Come First Served" (FCFS) or "First-in, First-out" (FIFO).** Situations like waiting for a haircut, ticket-booking counters follow FCFS discipline. Other disciplines include **"Last In First Out" (LIFO)** where last customer is serviced first, **"Service In Random Order" (SIRO)** in which the customers are serviced randomly irrespective of their arrivals. **"Priority service"** is when the customers are grouped in priority classes based on urgency. **"Preemptive Priority"** is the highest priority given to the customer who enters into the service, immediately, even if a customer with lower priority is in service. **"Non-preemptive priority"** is where the customer goes ahead in the queue, but will be served only after the completion of the current service.

4. ***The Number of Customers Allowed in the System:*** Some of the queuing processes allow the limitation to the capacity or size of the waiting room, so that the waiting line reaches a certain length, no additional customers is allowed to enter until space becomes available by a service completion. This type of situation means that there is a finite limit to the maximum queue size.

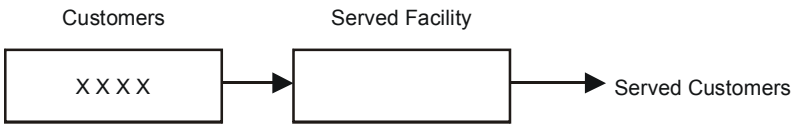
5. ***The Number of Service Channels:*** The more the number of service channels in the service facility, the greater the overall service rate of the facility. The combination of arrival rate and service rate is critical for determining the number of service channels. When there are a number of service channels available for service, then the arrangement of service depends upon the design of the system's service mechanism.

Parallel channels means, a number of channels providing identical service facilities so that several customers may be served simultaneously. Series channel means a customer go through successive ordered channels before service is completed. The arrangements of service facilities are illustrated in Figure 11.2. A queuing system is called a **one-server model,** i.e., when the system has only one server, and a **multi-server model** i.e., when the system has a number of parallel channels, each with one server.

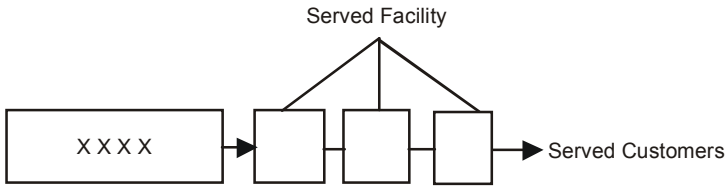
Notes

Figure 11.2: Arrangements of Service Facilities (a, b, c)

1. *Arrangement of service facilities in series*

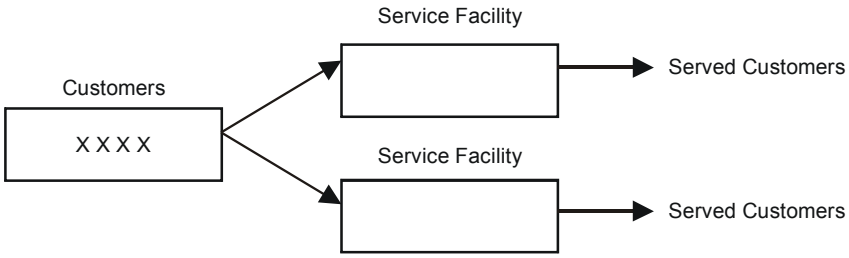


(a) Single Queue Single Server

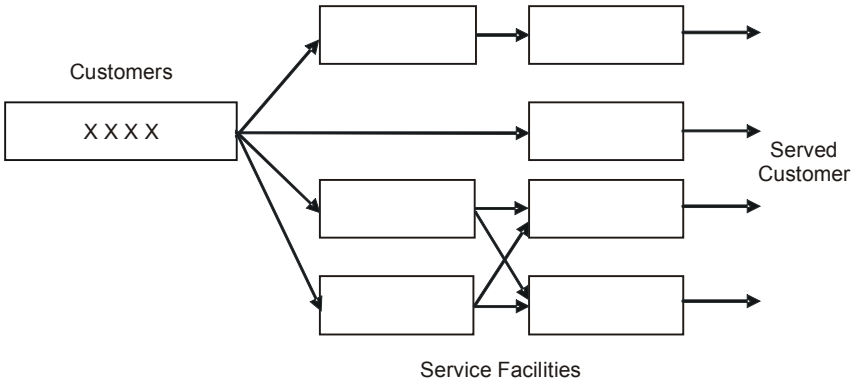


(b) Single Queue, Multiple Server

2. *Arrangement of Service facilities in Parallel*



3. *Arrangement of Mixed Service facilities*



11.1.2 Attitude of Customers

Patient Customer: Customer arrives at the service system, stays in the queue until served, no matter how much he has to wait for service.

Impatient Customer: Customer arrives at the service system, waits for a certain time in the queue and leaves the system without getting service due to some reasons like long queue before him.

Balking: Customer decides not to join the queue by seeing the number of customers already in service system.

Notes

Reneging: Customer after joining the queue, waits for some time and leaves the service system due to delay in service.

Jockeying: Customer moves from one queue to another thinking that he will get served faster by doing so.

Self Assessment

Multiple Choice Questions:

1. Which of the following characteristics apply to queueing system
   (a) Customer perception (b) Arrival process
   (c) Both a and b (d) none of the above
2. The cost of providing service in a queueing system decreases with
   (a) Decreased average waiting time in a queue
   (b) Decreased arrival time
   (c) Increased arrival rate
   (d) None of the above
3. Priority queue discipline may be classified as
   (a) Finite or infinite (b) Limited or unlimited
   (c) Preemptive or non preemptive (d) All of the above

11.2 Waiting Line Process

Waiting in lines is a part of our everyday life. Waiting in lines may be due to overcrowded, overfilling or due to congestion. Any time there is more customer demand for a service than can be provided, a waiting line forms. We wait in lines at the movie theater, at the bank for a teller, at a grocery store. Wait time depends on the number of people waiting before you, the number of servers serving line, and the amount of service time for each individual customer. Customers can be either humans or an object such as customer orders to be process, a machine waiting for repair. Mathematical analytical method of analyzing the relationship between congestion and delay caused by it can be modeled using Queuing analysis.

A waiting line process or queuing process is defined by two important elements:

- the population source of its customers and
- the process or service system.

The customer population can be considered as finite or infinite. The customer population is finite when the number of customers affects potential new customers for the service system already in the system. When the number of customers waiting in line does not significantly affect the rate at which the population generates new customers, the customer population is considered infinite. Customer behavior can change and depends on waiting line characteristics. In addition to waiting, a customer can choose other alternative.

When customer enters the waiting line but leaves before being serviced, process is called Reneging.

When customer changes one line to another to reduce wait time, process is called Jockeying. Balking occurs when customer do not enter waiting line but decides to come back latter.

Notes

Another element of queuing system is service system. The number of waiting lines, the number of servers, the arrangements of the servers, the arrival and service patterns, and the service priority rules characterize the service system. Queue system can have channels or multiple waiting lines.

Example: Single waiting line : bank counter, airline counters, restaurants, amusement parks.

In these examples multiple servers might serve customers. In the single line multiple servers has better performance in terms of waiting times and eliminates jockeying behavior than the system with a single line for each server. System serving capacity is a function of the number of service facilities and server proficiency. In waiting line system, the terms server and channel are used interchangeably. Waiting line systems are either single server or multiple servers.

Example:

1. ***Single server:*** gas station, food mart with single checkout counter, a theater with a single person selling tickets and controlling admission into the show.
2. ***Multiple server:*** gas stations with multiple gas pumps, grocery stores with multiple cashiers, multiple tellers in a bank.

Services require a single activity or services of activities called phases. In a single-phase system, the service is completed all at once, such as a bank transaction or grocery store checkout counter. In a multiphase system, the service is completed in a series of phases, such as at fast-food restaurant with ordering, pay, and pick-up windows.

The process of waiting line is characterized by rate at which customers arrive and are served by service system. Arrival rate specifies the average number of customers per time period. The service rate specifies the average number customers that can be serviced during a time period. The service rate governs capacity of the service system. It is the fluctuation in arrival and service patterns that causes wait in queuing system. A general waiting line system with parallel server is shown in Figure 8.1.

Self Assessment

Fill in the blanks:

4. A bank transaction or grocery store checkout counter is an example of ……..system.
5. …… specifies the average number of customers per time period.
6. The …… governs capacity of the service system.
7. It is the fluctuation in …… and ……..patterns that causes wait in queuing system.
8. When customer changes one line to another to reduce wait time, process is called …….

11.3 Poisson and Exponential Distributions

Both the Poisson and Exponential distributions play a prominent role in queuing theory.

Considering a problem of determining the probability of n arrivals being observed during a time interval of length t, where the following assumptions are made.

Notes

1. Probability that an arrival is observed during a small time interval (say of length v) is proportional to the length of interval. Let the proportionality constant be λ, so that the probability is λ v.
2. Probability of two or more arrivals in such a small interval is zero.
3. Number of arrivals in any time interval is independent of the number in non-overlapping time interval.

These assumptions may be combined to yield what probability distributions are likely to be, under Poisson distribution with exactly n customers in the system.

Suppose function P is defined as follows:

P (n customers during period t) = the probability that n arrivals will be observed in a time interval of length t

then, $$P(n, t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \quad (n = 0, 1, 2, \ldots\ldots\ldots\ldots) \qquad (1)$$

This is the Poisson probability distribution for the discrete random variable n, the number of arrivals, where the length of time interval, t is assumed to be given. This situation in queuing theory is called **Poisson arrivals.** Since the arrivals alone are considered (not departures), it is called a **pure birth process.**

The time between successive arrivals is called **inter-arrival time.** In the case where the number of arrivals in a given time interval has Poisson distribution, inter-arrival times can be shown to have the exponential distribution. If the inter-arrival times are independent random variables, they must follow an exponential distribution with density f(t) where,

$$f(t) = \lambda e^{-\lambda t} \quad (t > 0) \qquad (2)$$

Thus for Poisson arrivals at the constant rate λ per unit, the time between successive arrivals (inter-arrival time) has the exponential distribution. The average Inter - arrival time is denoted by $\bar{I}$.

By integration, it can be shown that $E(t) = \bar{I}/\lambda$ (3)

If the arrival rate λ = 30/hour, the average time between two successive arrivals are 1/30 hour or 2 minutes.

For example, in the following arrival situations, the average arrival rate per hour, λ and the average inter arrival time in hour, are determined.

1. One arrival comes every 15 minutes.

   Average arrival rate, $l = \frac{60}{15}$ = 4 arrivals per hour.

   Average inter arrival time $\bar{I}$ = 15 minutes = ¼ or 0.25 hour.
2. Three arrivals occur every 6 minutes.

   Average arrival rate, l = 30 arrivals per hour.

   Average Inter-arrival time, $\bar{I} = \frac{6}{3}$ = 2 minutes = $\frac{1}{30}$ or 0.33 hr.

Notes

3. Average interval between successive intervals is 0.2 hour.

Average arrival rate, $l = \frac{1}{0.2}$ = 5 arrivals per hour.

Average Inter-arrival time, $\bar{I}$ = 0.2 hour.

Similarly, in the following service situations, the average service rate per hour, μ and average service time in hours are determined.

1. One service is completed in 10 minutes.

Average service rate, $m = \frac{60}{10}$ = 6 services per hour.

Average service time, $\bar{S} = \frac{30}{4}$ = 10 minutes or 0.166 hour.

2. Number of customers served in 15 minutes is 4.

Average service rate, $m = \frac{4}{15} \times 60$ =16 services per hour.

Average services time, $\bar{S} = \frac{30}{4}$ = 3.75 mins or 0.0625 hour.

3. Average service time is 0.25 hour.

Average service rate, m = 4 services per hour.

Average service time $\bar{S}$ = 15 mins or 0.25 hour.

Example: In a factory, the machines break down and require service according to a Poisson distribution at the average of four per day. What is the probability that exactly six machines break down in two days?

Solution:

Given $\lambda = 4, n = 6, t = 2$

$P(n, t) = P(6, 4)$ when $l = 4$

we know, $$P(n, t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$P(6,2) = \frac{(4 \times 2)^6 e^{-4 \times 2}}{6!}$$

$$= \frac{8^6 e^{-8}}{720}$$

$$= 0.1221$$

Example: On an average, 6 customers arrive in a coffee shop per hour. Determine the probability that exactly 3 customers will reach in a 30 minute period, assuming that the arrivals follow Poisson distribution.

Notes

Solution:

Given, l = 6 customers / hour

t = 30 Minutes = 0.5 hour

n = 2

we know, $$P(n, t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$P(6,2) = \frac{(6 \times 0.5)^2 e^{-6\times 0.5}}{2!} = 0.22404$$

Similarly, when the time taken to serve different customers are independent, the probability that no more than t periods would be required to serve a customer is given by exponential distribution as follows:

p(not more than t time period) = $1 - e^{-\mu t}$ where μ = average service rate

Task A manager of a fast food restaurant observes that, an average of 9 customers are served by a waiter in a one-hour time period. Assuming that the service time has an exponential distribution, what is the probability that

1. A customer shall be free within 12 minutes.
2. A customer shall be serviced in more than 25 minutes.

11.4 Symbols and Notations

The symbols and notations used in queuing system are as follows:

n = Number of customers in the system (both waiting and in service).

λ = Average number of customers arriving per unit of time.

μ = Average number of customers being served per unit of time.

λ / μ = ρ, traffic intensity.

C = Number of parallel service channels (i.e., servers).

L_s = Average or expected number of customers in the system (both waiting and in service).

L_q = Average or expected number of customers in the queue.

W_s = Average waiting time in the system (both waiting and in service).

W_q = Average waiting time of a customer in the queue.

P_n = Time independent probability that there are n customers in the system (both waiting and in service).

$P_n(t)$ = Probability that there are n customers in the system at any time t (both waiting and in service).

Notes

Did u know? Different models in queueing theory are classified by using special notations described initially in 1953 in the form (a/b/b). Later A.M. Lee in 1966 added the symbols d and c to the Kendall notation. Now in the literature of queuing theory the standard format used to describe the main characteristics of parallel queues is as follows:

{(a/b/c) = (d/c)}

Where a = arrivals distribution

b = service time

c = number of service channels (servers)

d = maximum number of customers allowed in the system

e = queue discipline

11.5 Model 1: (MM1) : (α / FIFO)

Assumptions

This model is based on the following assumptions:

1. The arrivals follow Poisson distribution, with a mean arrival rate λ.
2. The service time has exponential distribution, average service rate μ.
3. Arrivals are infinite population α.
4. Customers are served on a First-in, First-out basis (FIFO).
5. There is only a single server.

System of Steady-state Equations

In this method, the question arises whether the service can meet the customer demand. This depends on the values of λ and μ.

If $\lambda \geq \mu$, i.e., if arrival rate is greater than or equal to the service rate, the waiting line would increase without limit. Therefore for a system to work, it is necessary that $\lambda < \mu$.

As indicated earlier, traffic intensity $\rho = \lambda / \mu$. This refers to the probability of time. The service station is busy. We can say that, the probability that the system is idle or there are no customers in the system, $P_0 = 1 - \rho$.

From this, the probability of having exactly one customer in the system is $P_1 = \rho P_0$.

Likewise, the probability of having exactly 2 customers in the system would be

$$P_3 = \rho P_1 = \rho^2 P_0$$

The probability of having exactly n customers in the system is

$$P_n = \rho^n P_0 = \rho^{n(1-\rho)} = (\lambda / \mu)^n P_0$$

Notes

The expected number of customers in the system is given by,

$$L_s = \sum_{n=1}^{\alpha} nP_n = \sum_{n=1}^{\alpha} n(1-\lambda/\mu)(\lambda/\mu)^n$$

$$= \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho} \qquad (2)$$

The expected number of customers in the queue is given by,

$$L_n = \sum_{n=1}^{\alpha} (n-1)P_n$$

$$= \sum_{n=1}^{\alpha} nP_n - \sum_{n=1}^{\alpha} P_n$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho} \qquad (3)$$

With an average arrival rate λ, the average time between the arrivals is $1/\lambda$. Therefore, the mean waiting time in queue, w_q is the product of the average time between the arrivals and the average queue length,

$$W_q = \left[\frac{1}{\lambda}\right]\left[\frac{1}{1-\rho}\right] \qquad (4)$$

$$= \left[\frac{1}{\lambda}\right]\left[\frac{\mu}{\mu-\lambda}\right]$$

Substituting $\left[\frac{\lambda^2}{\mu(\mu-\lambda)}\right] = \frac{\rho}{\mu-\lambda}$

Similarly the average waiting time in the system,

$$W_s = \left[\frac{1}{\lambda}\right]\left[\frac{\rho}{1-\rho}\right] \qquad (5)$$

putting $L_s = l\,(m - l)$,

we get $W_s = \frac{1}{\mu-\lambda}$

Queuing Equations

The evaluation of Model I is listed below:

1. ***Expected number of customers in the system:***

$$L_s = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho}$$

2. ***Expected number of customers in the queue:***

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}$$

Notes

3. ***Average waiting time in the system:***

$$W_s = \frac{1}{\mu - \lambda}$$

4. ***Average waiting time in the queue:***

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

5. ***Average waiting time for a customer:***

$$W(w / w > 0) = \frac{1}{\mu(1-\rho)} \text{ or } \frac{1}{\mu - \lambda}$$

6. ***Expected length of non-empty queue:***

$$L(m / m > 0) = \frac{\mu}{(\mu - \lambda)}$$

7. ***Probability that there are n customers in the system:***

$$P_n = \left[\frac{\lambda}{\mu}\right]^n P_0 = \left[\frac{\lambda}{\mu}\right]^n \left[1 - \frac{\lambda}{\mu}\right]$$

8. ***Probability that there is nobody in the system:***

$$P_0 = 1 - \frac{\lambda}{\mu}$$

9. ***Probability that there is at least one customer or queue is busy,***

$$P_b = 1 - P_0$$

10. ***Traffic intensity, or utilization factor,***

$$\rho = \frac{\lambda}{\mu}$$

Example: Consider a situation where the mean arrival rate (l) is one customer every 4 minutes and the mean service time (m) is 2½ minutes. Calculate the average number of customers in the system, the average queue length and the time taken by a customer in the system and the average time a customer waits before being served.

Solution:

Given, Average Arrival Rate λ = 1 customer every 4 minutes or 15 customers per hour

Average Service -Rate μ = 1 customer every 2½ minutes or 24 customers per hour

1. ***The average number of customers in the system:***

$$L_s = \frac{\lambda}{\mu - \lambda}$$

$$= \frac{15}{24 - 15} = 1.66 \text{ customers}$$

Notes

2. ***The average queue length:***

$$L_q = \left[\frac{\lambda}{\mu}\right]\left[\frac{\lambda}{\mu - \lambda}\right]$$

$$= \frac{15}{24} \times \frac{15}{24 - 15}$$

= 1.04 customers

3. ***The average time a customer spends in the system:***

$$W_s = \frac{1}{\mu - \lambda}$$

$$= \frac{1}{24 - 15}$$

= 0.11 × 60 = 6.66 minutes

4. ***The average time a customer waits before being served:***

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$= \frac{15}{24(24 - 15)}$$

= 0.069 × 60

= 4.16 minutes

Example: Trucks at a single platform weigh-bridge arrive according to Poisson probability distribution. The time required to weigh the truck follows an exponential probability distribution. The mean arrival rate is 12 trucks per day, and the mean service rate is 18 trucks per day. Determine the following:

1. What is the probability that no trucks are in the system?
2. What is the average number of trucks waiting for service?
3. What is the average time a truck waits for weighing service to begin?
4. What is the probability that an arriving truck will have to wait for service?

Solution:

Given λ = 12 trucks per days, μ = 18 trucks per day.

1. Probability that no trucks are waiting for service,

$$P_0 = 1 - \frac{\lambda}{\mu}$$

$$= 1 - \frac{12}{18}$$

= 0.3333 or 33.33%

Notes

2. Average number of trucks waiting for service,

$$L_q = \left[\frac{\lambda}{\mu}\right]\left[\frac{\lambda}{\mu - \lambda}\right]$$

$$= \left[\frac{12}{18}\right]\left[\frac{12}{18 - 12}\right]$$

= 1.33 trucks

3. Average time a truck waits for weighing service to begin,

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$= \frac{12}{18(18 - 12)}$$

= 0.1111 days or 53.3 minutes.

4. Probability that an arriving truck will have to wait for service,

$P_0 = 1 - P_0$

= 1 – 0.333

= 0.6667 or 66.67%

Tasks Solve the following questions:

1. A TV repairman finds that the time spent on his jobs has a exponential distribution with mean 30 minutes. If he repairs TV sets in the order in which they come in, and if the arrivals follow approximately Poisson distribution with an average rate of 10 per 8 hour day, what is the repairman's expected idle time each day? How many jobs are ahead of the average with the set just brought in?
2. Auto car service provides a single channel water wash service. The incoming arrivals occur at the rate of 4 cars per hour and the mean service rate is 8 cars per hour. Assume that arrivals follow a Poisson distribution and the service rate follows an exponential probability distribution. Determine the following measures of performance:
   (a) What is the average time that a car waits for water – wash to begin?
   (b) What is the average time a car spends in the system?
3. What is the average number of cars in the system?

Self Assessment

State true or false:

9. An important assumptions in Single server Queuing model 1 is that the customers are served on a First-in, First-out basis (FIFO).
10. If arrival rate is lesser than or equal to the service rate, the waiting line would increase without limit.

Notes

11. Traffic intensity in a system of steady state is given by $\rho = l / \mu$.

12. For Poisson arrivals at the constant rate l per unit, the time between successive arrivals (inter-arrival time) has the exponential distribution.

11.6 Summary

- Queuing Theory is a collection of mathematical models of various queuing systems.
- It is used extensively to analyze production and service processes exhibiting random variability in market demand (arrival times) and service time.
- Queues or waiting lines arise when the demand for a service facility exceeds the capacity of that facility, that is, the customers do not get service immediately upon request but must wait, or the service facilities stand idle and wait for customers.
- The type of queuing system a business uses is an important factor in determining how efficient the business is run.
- As the size of the population increases the world over, the number of queues and their queue length also increase.
- In the business world, more customers mean more business transactions.
- Out of the many ways to attract customers, an efficient queuing system plays a significant role as it reduces a customer's waiting time. The shorter waiting time makes customers happy, and in all probabilities, a happy customer will come back for business again.
- In a queuing system, the calling population is assumed to be infinite.
- This means that if a unit leaves the calling population and joins the waiting line or enters service, there will be no change in the arrival rate.
- The arrivals occur one at a time in a random order and once the customer joins the queuing system he will eventually receive the service.
- The arrival rate and services are modeled as variables that follow statistical distributions. If the arrival rate is greater than the service rate, the waiting line will grow without bound.
- Waiting line models that assume that customers arrive according to a Poisson probability distribution, and service times are described by an exponential distribution.
- The Poisson distribution specifies the probability that a certain number of customers will arrive in a given time period.
- The exponential distribution describes the service times as the probability that a particular service time will be less than or equal to a given amount of time.
- A waiting line priority rule determines which customer is served next. A frequently used priority rule is first-come, first-served.
- Other rules include best customers first, high-test profit customer first, emergencies first, and so on.
- Although each priority rule has merit, it is important to use the priority rule that best supports the overall organization strategy.
- The priority rule used affects the performance of the waiting line system.

Notes

- Basic single server model assumes customers are arriving at Poisson arrival rate with exponential service times, and first come, first serviced queue discipline, and infinite queue length, and infinite calling population.
- By adding additional resources to single server system either service rate can be increased or arrival rate at each server can be decreased with additional cost overhead.
- In Single server single-phase system, customer is served once completed.
- In single server queuing system wait time or performance of system depends on efficiency of serving person or service machine.
- Single server single-phase queuing system is most commonly automated system found in our regular life.

11.7 Keywords

Arrival Rate: The number of customers arriving at an average.

Calling Population: Source of customers or messages, the nature of customer numbers.

Queue: A waiting line.

Service Time: The time taken to serve a customer.

Waiting Line System: A system that consists of arrivals in queues, servers and waiting line structures.

11.8 Review Questions

1. What is the queuing theory?
2. Define arrival rate and service rate.
3. Explain the characteristics of MM1 queuing model.
4. Briefly explain Service Mechanism and Queue Discipline.
5. What is system of steady state?
6. What is multi-server system?
7. Give any four applications of MM1 queuing model.
8. How the time spent in the system is calculated?
9. What is queue discipline?
10. What is traffic intensity?

Answers: Self Assessment

1.	(c)	2.	(d)
3.	(c)	4.	Single Phase
5.	Arrival	6.	Service
7.	arrival, service	8.	jockeying
9.	True	10.	False
11.	True	12.	True

11.9 Further Readings

Notes

Books

D. Jerry Banks, John S. Carson, *Discrete-Event System Simulation*, Prentice Hall

K. Watkins, *Discrete Event Simulation in C*, McGraw-Hill.

Online links

www.waitinglinetheory.com

orms.czu.cz

www.usfca.edu

Notes

Unit 12: Critical Path Method and PERT

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Objectives

Introduction

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12.2 Rules in Constructing a Network

12.3 Critical Path Method (CPM)

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12.4 Project Evaluation Review Technique (PERT)

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Objectives

After studying this unit, you will be able to:

- Understand the importance of CPM and PERT techniques for project management
- Construct network diagrams
- Learn about the concept of time estimates, slack and float.
- Learn how to find critical paths.

Introduction

Any project involves planning, scheduling and controlling a number of interrelated activities with use of limited resources, namely, men, machines, materials, money and time. The projects may be extremely large and complex such as construction of a power plant, a highway, a shopping complex, ships and aircraft, introduction of new products and research and development projects. It is required that managers must have a dynamic planning and scheduling system to produce the best possible results and also to react immediately to the changing conditions and make necessary changes in the plan and schedule. A convenient analytical and visual technique of PERT and CPM prove extremely valuable in assisting the managers in managing the projects.

Both the techniques use similar terminology and have the same purpose. PERT stands for Project Evaluation and Review Technique developed during 1950s. The technique was developed and used in conjunction with the planning and designing of the Polaris missile project. CPM stands for Critical Path Method which was developed by DuPont Company and applied first to the construction projects in the chemical industry. Though both PERT and CPM techniques have similarity in terms of concepts, the basic difference is, PERT is used for analysis of project scheduling problems. CPM has single time estimate and PERT has three time estimates for activities and uses probability theory to find the chance of reaching the scheduled time.

Notes

Project management generally consists of three phases:

1. ***Planning:*** Planning involves setting the objectives of the project. Identifying various activities to be performed and determining the requirement of resources such as men, materials, machines, etc. The cost and time for all the activities are estimated, and a network diagram is developed showing sequential interrelationships (predecessor and successor) between various activities during the planning stage.
2. ***Scheduling:*** Based on the time estimates, the start and finish times for each activity are worked out by applying forward and backward pass techniques, critical path is identified, along with the slack and float for the non-critical paths.
3. ***Controlling:*** Controlling refers to analyzing and evaluating the actual progress against the plan. Reallocation of resources, crashing and review of projects with periodical reports are carried out.

12.1 CPM/PERT Network Components

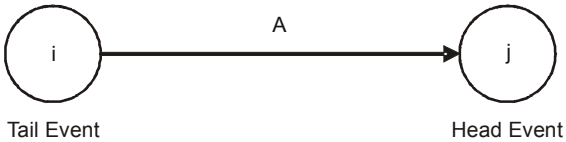
CPM/PERT networks contains two major components

1. ***Activity:*** An activity represents an action and consumption of resources (time, money, energy) required to complete a portion of a project. Activity is represented by an arrow, (Figure 12.1):

Figure 12.1: An Activity

2. ***Event:*** An event (or node) will always occur at the beginning and end of an activity. The event has no resources and is represented by a circle. The ith event and jth event are the tail event and head event respectively, (Figure 12.2).

Figure 12.2: An Event



Merge and Burst Events

One or more activities can start and end simultaneously at an event (Figure 12.3 a, b).

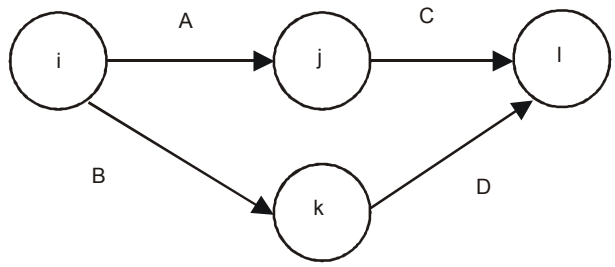
Figure 12.3

Notes

Preceding and Succeeding Activities

Activities performed before given events are known as preceding activities (Figure 12.4), and activities performed after a given event are known as succeeding activities.

Figure 12.4: Preceding and Succeeding Activities

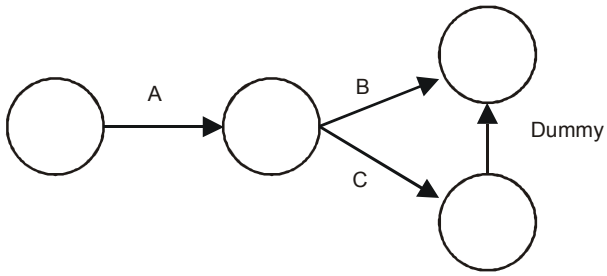


Activities A and B precede activities C and D respectively.

Dummy Activity

An imaginary activity which does not consume any resource and time is called a dummy activity. Dummy activities are simply used to represent a connection between events in order to maintain a logic in the network. It is represented by a dotted line in a network, see Figure 12.5.

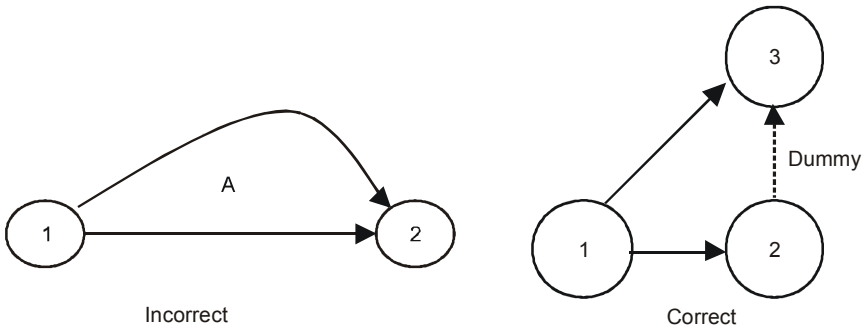
Figure 12.5: Dummy Activity



Errors to be avoided in Constructing a Network

1. Two activities starting from a tail event must not have a same end event. To ensure this, it is absolutely necessary to introduce a dummy activity, as seen in Figure 12.6.

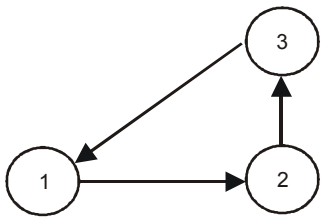
Figure 12.6: Correct and Incorrect Activities



Notes

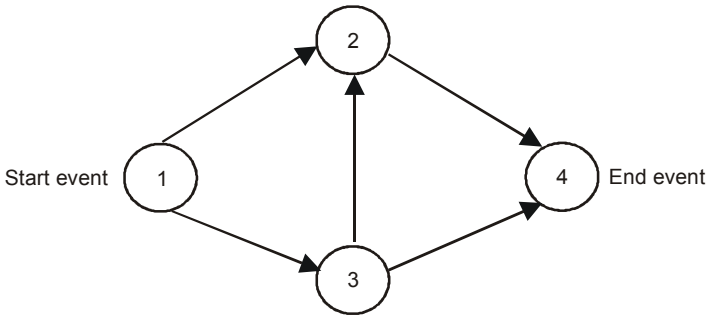
2. Looping error should not be formed in a network, as it represents performance of activities repeatedly in a cyclic manner, as shown below in Figure 12.7.

Figure 12.7: Looping Error



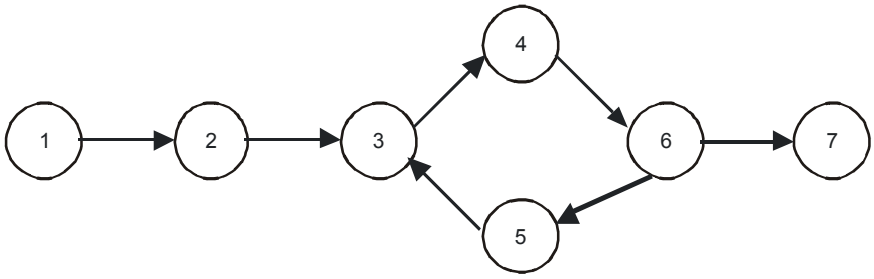
3. In a network, there should be only one start event and one ending event as shown below, in Figure 12.8.

Figure 12.8: Only One Start and End Event



4. The direction of arrows should flow from left to right avoiding mixing of direction as shown in Figure 12.9.

Figure 12.9: Wrong Direction of Arrows



Incorrect

Self Assessment

Give one word for the following:

1. The shortest time taken to complete an activity.
2. The technique in which the time estimates are assumed to be known with certainty.
3. An imaginary activity that does not consume any resource and time
4. Activities that must be used only if it is necessary to reduce the complexity of the network.

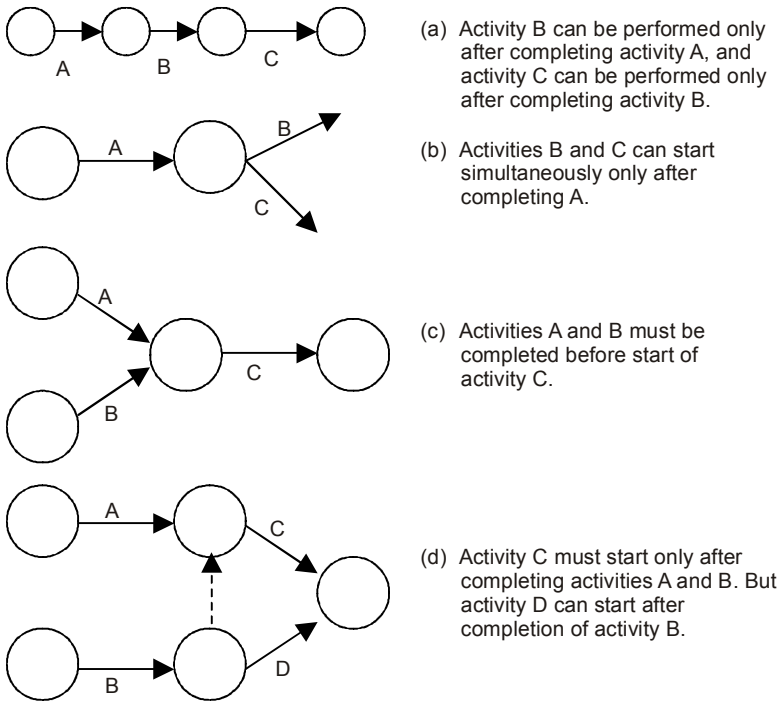
Notes

12.2 Rules in Constructing a Network

1. No single activity can be represented more than once in a network. The length of an arrow has no significance.
2. The event numbered 1 is the start event and an event with highest number is the end event. Before an activity can be undertaken, all activities preceding it must be completed. That is, the activities must follow a logical sequence (or interrelationship) between activities.
3. In assigning numbers to events, there should not be any duplication of event numbers in a network.
4. Dummy activities must be used only if it is necessary to reduce the complexity of a network.
5. A network should have only one start event and one end event.

Some conventions of network diagram is shown in Figure 12.10 (a), (b), (c), (d) below:

Figure 12.10 (a), (b), (c), (d): Some Conventions followed in Making Network Diagrams



Notes **Procedure for Numbering the Events using Fulkerson's Rule**

Step 1: Number the start or initial event as 1.

Step 2: From event 1, strike off all outgoing activities. This would have made one or more events as initial events (event which do not have incoming activities). Number that event as 2.

Step 3: Repeat step 2 for event 2, event 3 and till the end event. The end event must have the highest number.

Notes

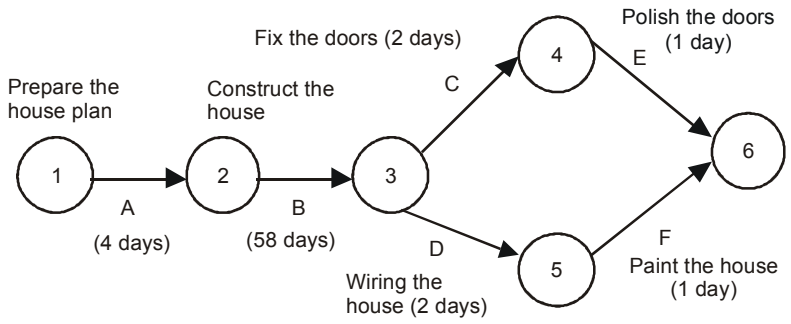
Example: Draw a network for a house construction project. The sequence of activities with their predecessors are given in Table 12.1, below.

Table 12.1: Sequence of Activities for House Construction Project

Name of the activity	Starting and finishing event	Description of activity	Predecessor	Time duration (days)
A	(1,2)	Prepare the house plan	--	4
B	(2,3)	Construct the house	A	58
C	(3,4)	Fix the door/windows	B	2
D	(3,5)	Wiring the house	B	2
E	(4,6)	Paint the house	C	1
F	(5,6)	Polish the doors/ windows	D	1

Solution:

Figure 12.11: Network Diagram Representing House Construction Project



The network diagram in Figure 12.11 shows the procedure relationship between the activities. Activity A (preparation of house plan), has a start event 1 as well as an ending event 2. Activity B (Construction of house) begins at event 2 and ends at event 3. The activity B cannot start until activity A has been completed. Activities C and D cannot begin until activity B has been completed, but they can be performed simultaneously. Similarly, activities E and F can start only after completion of activities C and D respectively. Both activities E and F finish at the end of event 6.

Example: Consider the project given in Table 12.2 and construct a network diagram.

Table 12.2: Sequence of Activities for Building Construction Project

Activity	Description	Predecessor
A	Purchase of Land	-
B	Preparation of building plan	-
C	Level or clean the land	A
D	Register and get approval	A, B
E	Construct the building	C
F	Paint the building	D

Notes

Solution:

The activities C and D have a common predecessor A. The network representation shown in Figure 12.12 (a), (b) violates the rule that no two activities can begin and end at the same events. It appears as if activity B is a predecessor of activity C, which is not the case. To construct the network in a logical order, it is necessary to introduce a dummy activity as shown in Figure 12.13.

Figure 12.12: Network Representing the Error

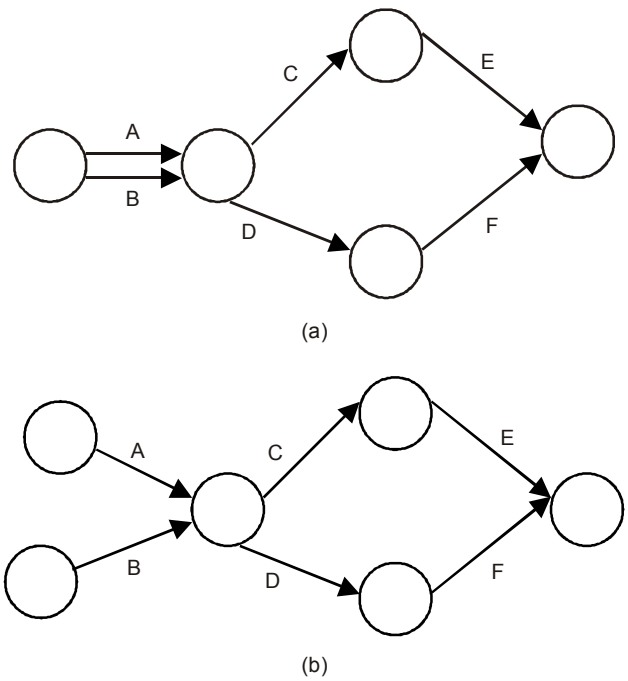
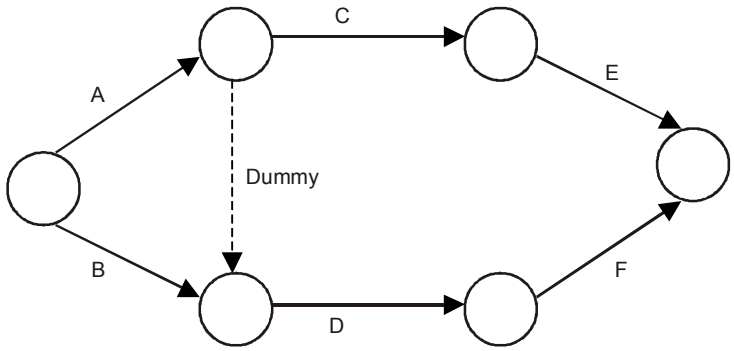


Figure 12.13: Correct Representation of Network using Dummy Activity



Example: Construct a network for a project whose activities and their predecessor relationship are given in Table 12.3.

Notes

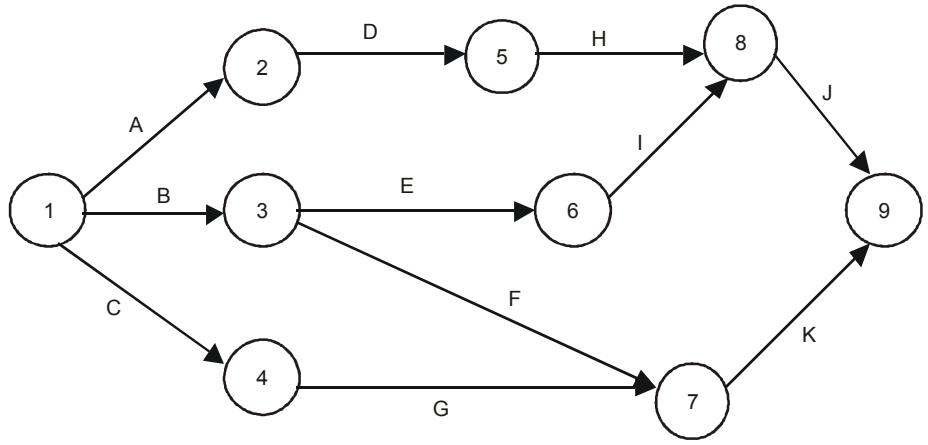
Table 12.3: Activity Sequence for a Project

Activity	A	B	C	D	E	F	G	H	I	J	K
Predecessor	-	-	-	A	B	B	C	D	E	H, I	F, G

Solution:

The network diagram for the given problem is shown in Figure 12.14 with activities A, B and C starting simultaneously.

Figure 12.14: Network Diagram



Task Draw a network diagram for a project given in Table.

Activity	A	B	C	D	E	F	G	H	I	J	K	L
Immediate Predecessor	-	A	B	A	D	C, E	D	D	H	H	F, H	G, J

12.3 Critical Path Method (CPM)

The critical path for any network is the longest path through the entire network. Since all activities must be completed to complete the entire project, the length of the critical path is also the shortest time allowable for completion of the project. Thus if the project is to be completed in that shortest time, all activities on the critical path must be started as soon as possible. These activities are called **critical activities**. If the project has to be completed ahead of the schedule, then the time required for at least one of the critical activity must be reduced. Further, any delay in completing the critical activities will increase the project duration.

The activity, which does not lie on the critical path, is called **non-critical activity.** These non-critical activities may have some slack time. The slack is the amount of time by which the start of an activity may be delayed without affecting the overall completion time of the project. But a critical activity has no slack. To reduce the overall project time, it would require more resources (at extra cost) to reduce the time taken by the critical activities to complete.

Notes

12.3.1 Time Estimates: Earliest Time and Latest Time

Before the critical path in a network is determined, it is necessary to find the earliest and latest time of each event to know the earliest expected time (T_E) at which the activities originating from the event can be started and to know the latest allowable time (T_L) at which activities terminating at the event can be completed.

Forward Pass Computations (to calculate Earliest, Time T_E)

Procedure

Step 1: Begin from the start event and move towards the end event.

Step 2: Put $T_E = 0$ for the start event.

Step 3: Go to the next event (i.e. node 2) if there is an incoming activity for event 2, add calculate T_E of previous event (i.e. event 1) and activity time. Note: If there are more than one incoming activities, calculate T_E for all incoming activities and take the maximum value. This value is the T_E for event 2.

Step 4: Repeat the same procedure from step 3 till the end event.

Backward Pass Computations (to calculate Latest Time T_L)

Procedure

Step 1: Begin from end event and move towards the start event. Assume that the direction of arrows is reversed.

Step 2: Latest Time T_L for the last event is the earliest time. T_E of the last event.

Step 3: Go to the next event, if there is an incoming activity, subtract the value of T_L of previous event from the activity duration time. The arrived value is T_L for that event. If there are more than one incoming activities, take the minimum T_E value.

Step 4: Repeat the same procedure from step 2 till the start event.

Determination of Float and Slack Times

As discussed earlier, the non-critical activities have some slack or float. The ***float*** of an activity is the amount of time available by which it is possible to delay its completion time without extending the overall project completion time.

For an activity $i = j$, let

t_{ij} = duration of activity

T_E = earliest expected time

T_L = latest allowable time

ES_{ij} = earliest start time of the activity

EF_{ij} = earliest finish time of the activity

LS_{ij} = latest start time of the activity

LF_{ij} = latest finish time of the activity

Notes

Total float TF_{ij}: The total float of an activity is the difference between the latest start time and the earliest start time of that activity.

$$TF_{ij} = LS_{ij} - ES_{ij} \quad (1)$$

or

$$TF_{ij} = (T_L - T_E) - t_{ij} \quad (2)$$

Free Float FF_{ij}: The time by which the completion of an activity can be delayed from its earliest finish time without affecting the earliest start time of the succeeding activity is called free float.

$$FF_{ij} = (E_j - E_i) - t_{ij} \quad (3)$$

$$FF_{ij} = \text{Total float} - \text{Head event slack}$$

Independent Float IF_{ij}: The amount of time by which the start of an activity can be delayed without affecting the earliest start time of any immediately following activities, assuming that the preceding activity has finished at its latest finish time.

$$IF_{ij} = (E_j - L_i) - t_{ij} \quad (4)$$

$$IF_{ij} = \text{Free float} - \text{Tail event slack}$$

Where tail event slack = $L_i - E_i$

Caution The negative value of independent float is considered to be zero.

Critical Path: After determining the earliest and the latest scheduled times for various activities, the minimum time required to complete the project is calculated. In a network, among various paths, the longest path which determines the total time duration of the project is called the ***critical path.*** The following conditions must be satisfied in locating the critical path of a network.

An activity is said to be critical only if both the conditions are satisfied.

1. $T_L - T_E = 0$
2. $T_{Lj} - t_{ij} - T_{Ej} = 0$

Example: A project schedule has the following characteristics as shown in Table 12.4

Table 12.4: Project Schedule

Activity	Name	Time	Activity	Name	Time (days)
1-2	A	4	5-6	G	4
1-3	B	1	5-7	H	8
2-4	C	1	6-8	I	1
3-4	D	1	7-8	J	2
3-5	E	6	8-10	K	5
4-9	F	5	9-10	L	7

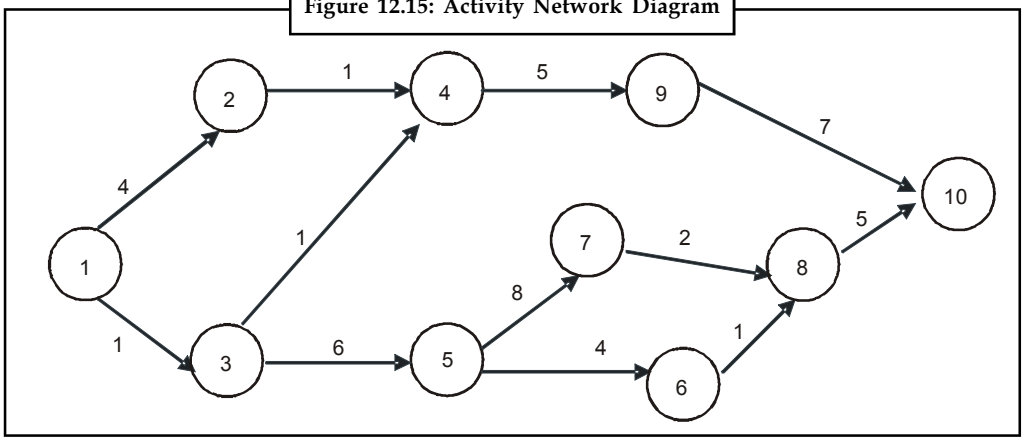
1. Construct a network diagram.
2. Compute T_E and T_L for each activity.
3. Find the critical path.

Notes

Solution:

1. From the data given in the problem, the activity network is constructed as shown in Figure 5.6.

Figure 12.15: Activity Network Diagram



2. To determine the critical path, compute the earliest, time T_E and latest time T_L for each of the activity of the project. The calculations of T_E and T_L are as follows:

To calculate T_E for all activities,

T_{E1} = 0

T_{E2} = $T_{E1} + t_{1, 2} = 0 + 4 = 4$

T_{E3} = $T_{E1} + t_{1, 3} = 0 + 1 = 1$

T_{E4} = max ($T_{E2} + t_{2, 4}$ and $T_{E3} + t_{3, 4}$)

= max (4 + 1 and 1 + 1) = max (5, 2)

= 5 days

T_{E5} = $T_{E3} + t_{3, 6} = 1 + 6 = 7$

T_{E6} = $T_{E5} + t_{5, 6} = 7 + 4 = 11$

T_{E7} = $T_{E5} + t_{5, 7} = 7 + 8 = 15$

T_{E8} = max ($T_{E6} + t_{6, 8}$ and $T_{E7} + t_{7, 8}$)

= max (11 + 1 and 15 + 2) = max (12, 17)

= 17 days

T_{E9} = $T_{E4} + t_{4, 9} = 5 + 5 = 10$

T_{E10} = max ($T_{E9} + t_{9, 10}$ and $T_{E8} + t_{8, 10}$)

= max (10 + 7 and 17 + 5) = max (17, 22)

= 22 days

Notes

To calculate T_L for all activities

T_{L10} = T_{E10} = 22

T_{L9} = $T_{E10} - t_9$, 10 = 22 – 7 = 15

T_{L8} = $T_{E10} - t_{8, 10}$ = 22 – 5 = 17

T_{L7} = $T_{E8} - t_{7, 8}$ = 17 – 2 = 15

T_{L6} = $T_{E8} - t_{6, 8}$ = 17 – 1 = 16

T_{L5} = min ($T_{E6} - t_{5, 6}$ and $T_{E7} - t_{5, 7}$)

= min (16 – 4 and 15 -8) = min (12, 7)

= 7 days

T_{L4} = $T_{L9} - t_{4, 9}$ = 15 – 5 =10

T_{L3} = min ($T_{L4} - t_{3, 4}$ and $T_{L5} - t_{3, 5}$)

= min (10 – 1 and 7 – 6) = min (9, 1)

= 1 day

T_{L2} = $T_{L4} - t_{2, 4}$ = 10 – 1 = 9

T_{L1} = Min ($T_{L2} - t_{1, 2}$ and $T_{L3} - t_{1, 3}$)

= Min (9 – 4 and 1 – 1) = 0

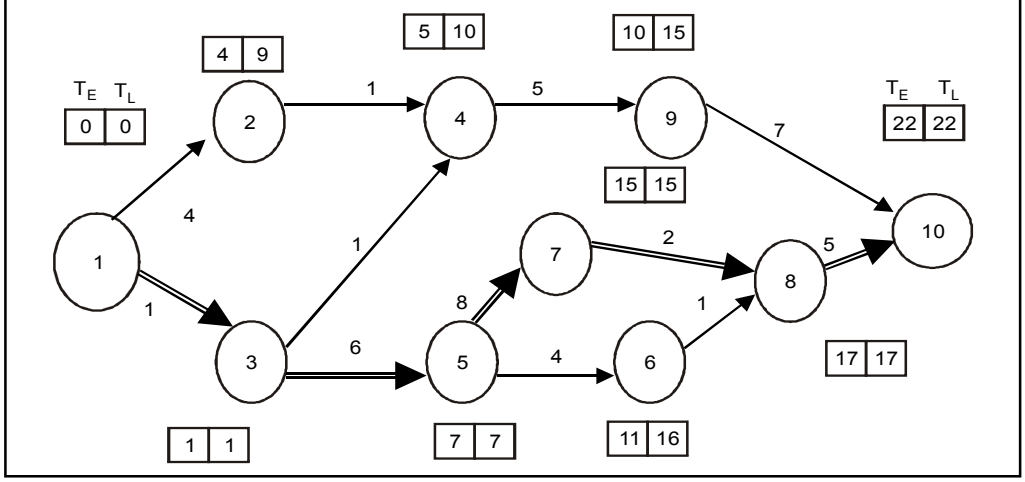
Table 12.5: Various Activities and their Floats

Activity	Activity Name	Normal Time	Earliest Time		Latest Time		Total Float
			Start	Finish	Start	Finish	
1-2	A	4	0	4	5	9	5
1-3	B	1	0	1	0	1	0
2-4	C	1	4	5	9	10	5
3-4	D	1	1	2	9	10	8
3-5	E	6	1	7	1	7	0
4-9	F	5	5	10	10	15	5
5-6	G	4	7	11	12	16	5
5-7	H	8	7	15	7	15	0
6-8	I	1	11	12	16	17	5
7-8	J	2	15	17	15	17	0
8-10	K	5	17	22	19	22	0
9-10	L	7	10	17	15	22	5

3. From the table 12.5, we observe that the activities 1 – 3, 3 – 5, 5 – 7,7 – 8 and 8 – 10 are critical activities as their floats are zero.

Notes

Figure 12.16: Critical Path of the Project



The critical path is 1-3-5-7-8-10 (shown in double line in Figure 12.16) with the project duration of 22 days.

Example: The following Table 12.6 gives the activities in construction project and time duration.

Table 12.6: Project Schedule with Time Duration

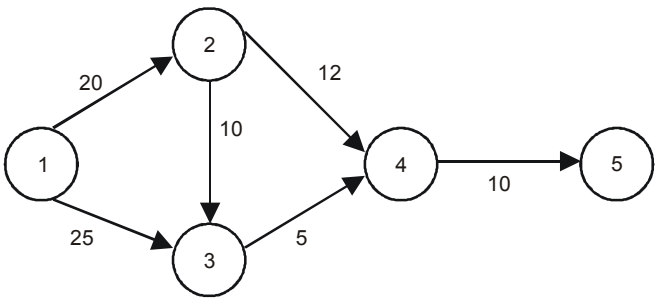
Activity	Preceding Activity	Normal time (days)
1-2	-	20
1-3	-	25
2-3	1-2	10
2-4	1-2	12
3-4	1-3,2-3	5
4-5	2-4,3-4	10

1. Draw the activity network of the project.
2. Find the total float and free float for each activity.

Solution:

1. From the activity relationship given, the activity network is shown in Figure 12.17 below:

Figure 12.17: Activity Network Diagram



Notes

2. The total and free floats for each activity are calculated as shown in Table 12.7.

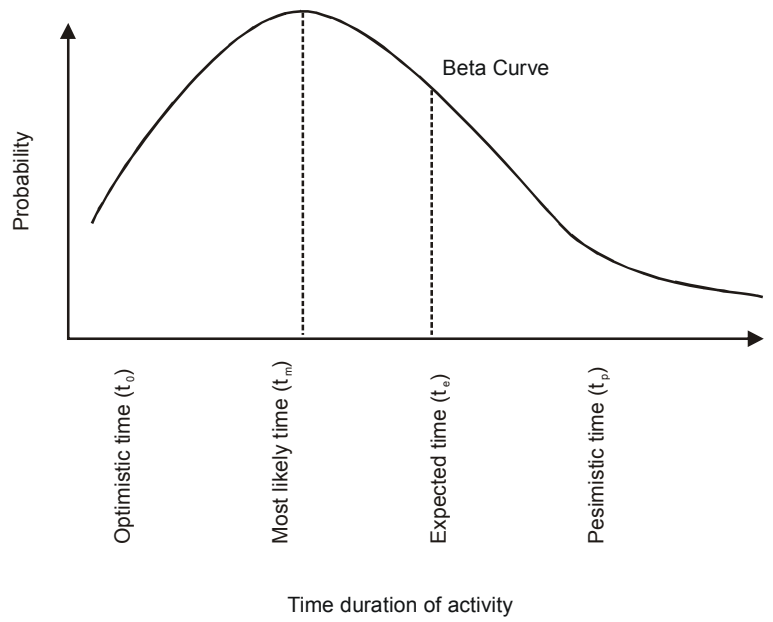
Table 12.7: Calculation of Total and Free Floats

Activity	Normal time (days)	Earliest Time		Latest Time		Float	
		Start	Finish	Start	Finish	Total	Free
1-2	20	0	20	0	20	0	0
1-3	25	0	25	5	30	5	5
2-3	10	20	30	20	30	0	0
2-4	12	20	32	23	35	3	3
3-4	5	30	35	30	35	0	0
4-5	10	35	45	35	45	0	0

12.4 Project Evaluation Review Technique (PERT)

In the critical path method, the time estimates are assumed to be known with certainty. In certain projects like research and development, new product introductions, it is difficult to estimate the time of various activities. Hence PERT is used in such projects with a probabilistic method using three time estimates for an activity, rather than a single estimate, as shown in Figure 12.18.

Figure 12.18: PERT using Probabilistic Method with 3 Time Estimates



Optimistic time t_0: It is the shortest time taken to complete the activity. It means that if everything goes well then there is more chance of completing the activity within this time.

Most likely time t_m: It is the normal time taken to complete an activity, if the activity were frequently repeated under the same conditions.

Pessimistic time t_p: It is the longest time that an activity would take to complete. It is the worst time estimate that an activity would take if unexpected problems are faced.

Notes

Taking all these time estimates into consideration, the expected time of an activity is arrived at.

The average or mean (t_a) value of the activity duration is given by,

$$T_a = \frac{t_0 + 4t_m + t_p}{6} \tag{5}$$

The variance of the activity time is calculated using the formula,

$$\sigma^2 = \left(\frac{t_p - t_0}{6}\right)^2 \tag{6}$$

Probability for Project Duration

The probability of completing the project within the scheduled time (T_s) or contracted time may be obtained by using the standard normal deviate where T_e is the expected time of project completion.

$$Z_0 = \frac{T_s - T_e}{\sqrt{\Sigma\sigma^2 \text{ in critical path}}} \tag{7}$$

Probability of completing the project within the scheduled time is,

$$P(T \leq T_s) = P(Z \leq Z_0) \text{ (from normal tables)} \tag{8}$$

Example: An R&D project has a list of tasks to be performed whose time estimates are given in the Table 12.8, as follows.

Time expected for each activity is calculated using the formula (5):

$$T_a = \frac{t_0 + 4tm + tp}{6}$$

$$= \frac{4 + 4(6) + 8}{6} = \frac{36}{6} = 6 \text{ days for activity A}$$

Similarly, the expected time is calculated for all the activities.

The variance of activity time is calculated using the formula (6).

$$\sigma_1^2 = \left(\frac{t_p - t_0}{6}\right)^2$$

$$= \left(\frac{8-4}{6}\right)^2 = 0.444$$

Similarly, variances of all the activities are calculated. Construct a network diagram and calculate the time earliest, T_E and time Latest T_L for all the activities.

Notes

Figure 12.19: Network Diagram

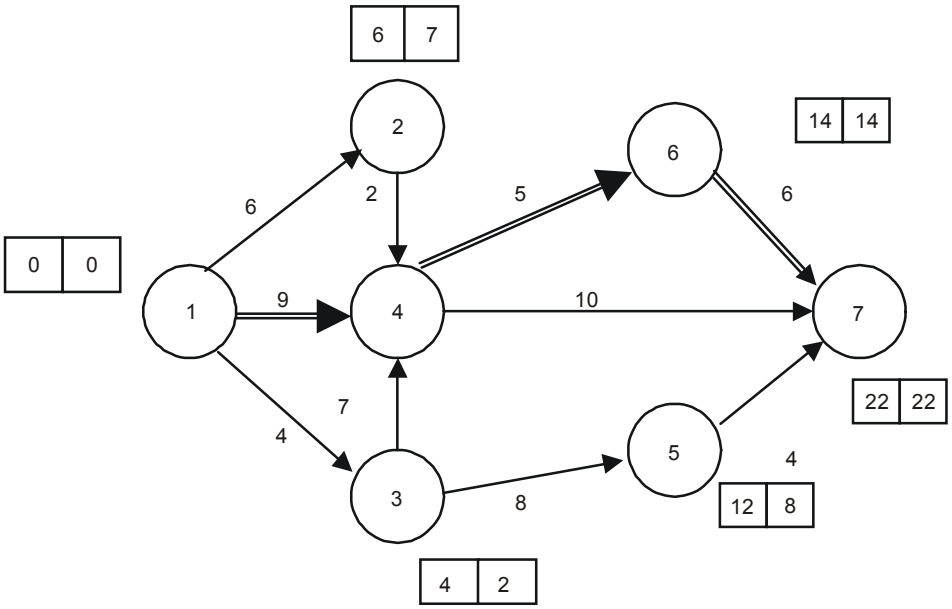


Table 12.8: Time Estimates for R&D Project

Activity I–j	Activity Name	T_0	t_m (in days)	t_p
1-2	A	4	6	8
1-3	B	2	3	10
1-4	C	6	8	16
2-4	D	1	2	3
3-4	E	6	7	8
3-5	F	6	7	14
4-6	G	3	5	7
4-7	H	4	11	12
5-7	I	2	4	6
6-7	J	2	6	10

1. Draw the project network.
2. Find the critical path.
3. Find the probability that the project is completed in 19 days. If the probability is less that 20%, find the probability of completing it in 24 days.

Notes

Solution:

Calculate the time average t_a and variances of each activity as shown in Table 12.9.

Table 12.9: T_e & σ^2 Calculated

Activity	T_o	T_m	T_p	T_a	σ^2
1-2	4	6	8	6	0.444
1-3	2	3	10	4	1.777
1-4	6	8	16	9	2.777
2-4	1	2	3	2	0.111
3-4	6	7	8	7	0.111
3-5	6	7	14	8	1.777
4-6	3	5	7	5	0.444
4-7	4	11	12	10	1.777
5-7	2	4	6	4	0.444
6-7	2	9	10	8	1.777

From the network diagram Figure 12.19, the critical path is identified as 1-4, 4-6, 6-7, with a project duration of 22 days.

The probability of completing the project within 19 days is given by, $P(Z \leq Z_0)$

To find Z_0,

$$Z_0 = \left(\frac{T_s - T_e}{\sqrt{\Sigma\sigma \text{ in critical path}}} \right)$$

$$= \left(\frac{19 - 22}{\sqrt{2.777 + 0.444 + 1.777}} \right)$$

$$= \left(\frac{-3}{\sqrt{5}} \right) = -1.3416 \text{ days}$$

we know, $P(Z < Z_0) = 0.5 - \Psi(1.3416)$ (from normal tables, $\Psi(1.3416) = 0.4099$)

$= 0.5 - 0.4099$

$= 0.0901$

$= 9.01\%$

Thus, the probability of completing the R&D project in 19 days is 9.01%. Since the probability of completing the project in 19 days is less than 20%, we find the probability of completing if in 24 days.

$$Z_0 = \frac{T_s - T_e}{\sqrt{\Sigma\sigma \text{ in critical path}}}$$

$$= \left(\frac{24 - 22}{\sqrt{5}} \right) = \left(\frac{2}{\sqrt{5}} \right) = 0.8944 \text{ days}$$

Notes

$P(Z \leq Z_0) = 0.5 - \Psi(0.8944)$ (from normal tables, $\Psi(0.8944) = 0.3133$)

$= 0.5 + 0.3133$

$= 0.8133$

$= 81.33\%$

Self Assessment

Fill in the blanks:

5. A ..... has no slack.
6. The ..... of an activity is the amount of time available by which it is possible to delay its completion time without extending the overall project completion time.
7. ..... refers to analyzing and evaluating the actual progress against the plan.
8. Project Management consists of ..... phases.
9. An ..... represents an action and consumption of resources required to complete a portion of a project.
10. ..... is the method to analyse the involved tasks in completing a given project.
11. Planning can be explained as the process of ..... and ..... a plan.
12. The probability of completing the project within the scheduled time or contracted time may be obtained by using the .....
13. ..... analysis is the general time given to certain specific techniques which can be used for the planning, management and control of projects.
14. Pessimistic time is the ..... time an activity would take to complete.
15. ..... time is the normal time taken to complete an activity.

12.5 Summary

- Network analysis is the general name given to certain specific techniques which can be used for the planning, management and control of projects.
- Critical Path Method (CPM), is a procedure for using network analysis to identify those tasks which are on the critical path i.e. where any delay in the completion of these tasks will lengthen the project timescale, unless action is taken.
- For all tasks off the critical path, a degree of tolerance is possible (e.g., late start, late completion, early start, etc.).
- The CPM formally identifies tasks which must be completed on time for the whole project to be completed on time Identifies which tasks can be delayed for a while if resource needs to be reallocated to catch up on missed tasks.
- It helps you to identify the minimum length of time needed to complete a project.
- The CPM determines both the early start and the late start date for each activity in the schedule.
- PERT is a method to analyze the involved tasks in completing a given project, especially the time needed to complete each task, and identifying the minimum time needed to complete the total project.

Notes

- It is more of an event-oriented technique rather than start- and completion-oriented, and is used more in projects where time, rather than cost, is the major factor.
- It is applied to very large-scale, one-time, complex, non-routine infrastructure and Research and Development projects.
- Program Evaluation and Review Technique (PERT) takes the CPM network and adds distributions to represent the activity times of the project.

12.6 Keywords

Controlling: The process of analyzing and evaluating the actual progress against the plan.

Float: The amount of time available by which it is possible to delay its completion time without extending the overall project completion time.

Planning: Process of creating and maintaining a plan.

Scheduling: The process of deciding how to commit resources between the varieties of possible tasks.

Slack: The amount of time that a task in a project network can be delayed without causing a delay to subsequent tasks and project completion date.

12.7 Review Questions

1. Prepare a network diagram for constructing a 5 floor apartment. The major activities of the project are given as follows:

Activity	Description	Immediate Predecessor
A	Selection of site	-
B	Preparation of drawings	-
C	Arranging the for finance	A
D	Selection of contractor	A
E	Getting approval from Govt	A
F	Laying the foundation	E
G	Start construction	D, F
H	Advertise in newspaper	B, C
I	Allocation of tenants	G, H

2. For the problem No. 1 the time estimates in days are given. Determine the Time earliest and Time latest, and the critical activities

Activity	A	B	C	D	E	F	G	H	I
Time (days)	3	5	7	2	5	20	60	2	10

3. An assembly having the following sequence of activities given along with their predecessor in the table below. Draw a network diagram for the assembly.

Notes

Activity	Description	Predecessor
A	Pick bolt & washer	-
B	Insert washer in screw	A
C	Fix the bolt in flange	A
D	Screw the nut with bolt	B, C
E	Pick the spanner	D
F	Tighten the nut	E
G	Place the assembly apart	F

4. Draw a network diagram for the project:

Activity	A	B	C	D	E	F	G	H	I	J
Predecessor	-	A	B	B	B	C	C	F, G	D, E, F	I

5. Determine the critical path and project duration for the following project:

Activity	Immediate Predecessor	Time (days)
A	-	3
B	-	7
C	A	4
D	B	2
E	C,D	5
F	A	6
G	E,F	3

6. A national conference is planned in a college. The activities are listed down along with their predecessors and time taken. Prepare a network diagram and determine the critical activities.

Activity	Description	Immediate Predecessor	Duration (days)
A	Confirm lead speaker and topic	-	5
B	Prepare brochure	-	1
C	Send letters to other speakers	B	2
D	Get confirmation from speakers	C	5
E	Send letters to participants	C,D	2
F	Obtain travel plans from speakers	D	2
G	Arrange for accommodation for speakers	F	1
H	Get handouts from speakers	F	4
I	Finalize registrations	G,H	10
J	Arrange hall and AV	I	1
K	Conduct of programme	J	1

Notes

7. Consider the following project with the list of activities:

Activity	Predecessors	Duration (months)
A	-	1
B	A	3
C	B	4
D	B	3
E	B	3
F	C	4
G	D,E	5
H	F	1
I	G,H	4
J	I	3
K	I	4
L	J	3
M	K	5
N	L	5

(a) Construct the project network diagram.

(b) Compute the earliest start time and earliest finish time.

(c) Find the latest start and latest finish time.

(d) Find the slack for each activity.

(e) Determine the critical path and project duration.

8. You are alone at home and have to prepare a bread sandwich for yourself. The preparation activities and time taken are given in the table below:

Task	Description	Predecessor	Time (minutes)
A	Purchase of bread	-	20
B	Take cheese and apply on bread	A	5
C	Get onions from freezer	A	1
D	Fry onions with pepper	B,C	6
E	Purchase sauce for bread	-	15
F	Toast Bread	B,C	4
G	Assemble bread and fried onions	F	2
H	Arrange in plate	G	1

(a) Determine the critical activities and preparation time for tasks given in table.

(b) Find the earliest time and latest time for all activities.

(c) While purchasing sauce, you met a friend and spoke to him for 6 minutes. Did this cause any delay in preparation?

9. An amusement park is planned at a suitable location. The various activities are listed with time estimates. Determine the critical path. Also, find whether the amusement park can be opened for public within 35 days from the start of the project work.

Notes

Activity:	A	B	C	D	E	F	G	H	I	J	K
Time (days):	9	6	2	7	10	3	6	1	7	2	5

The predecessor activities are given below:

Activity	Predecessor
A	-
B	-
C	A
D	A
E	C
F	B
G	E,F
H	D
I	H,E
J	I
K	G

10. Draw the network from the following activity and find the critical path and total duration of project.

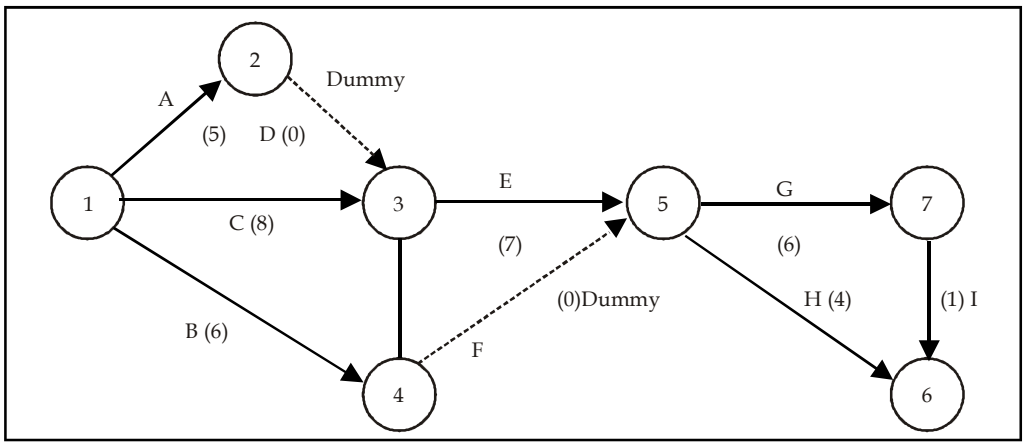
Activity	Duration (days)
1-2	5
1-3	3
1-4	6
2-3	8
2-5	7
3-5	2
4-5	6

11. Draw a network diagram and determine the project duration.

Activity	Duration (weeks)
1-2	2
1-4	4
1-3	7
2-5	6
3-4 (Dummy)	0
4-6	6
3-6	8
5-7	10
5-6	9
5-8	2
6-7	6
7-9	2
8-9	5

Notes

12. Determine the critical path and project duration for the network given.



13. For the PERT problem find the critical path and project duration. What is the probability that the project will be completed in 25 days?

Activity	Predecessor	Time		
		Optimistic	Most likely	Pessimistic
A	-	2	5	14
B	-	1	10	12
C	A	0	0	6
D	A	1	4	7
E	C	3	10	15
F	D	3	5	7
G	B	1	2	3
H	E,F	5	10	15
I	G	3	6	9

14. The following table lists the jobs of a network along with their estimates.

Job	Duration (days)		
	Optimistic	Most likely	Pessimestic
1-2	3	6	15
1-6	2	5	14
2-3	6	12	30
2-4	2	5	8
3-5	5	11	17
4-5	3	6	15
6-7	3	9	27
5-8	1	4	7
7-8	4	19	28

(a) Draw the project network diagram.

(b) Calculate the length and variance of the critical path.

(c) What is the probability that the jobs on the critical path can be completed in 41 days?

Notes

15. Draw the network for the following project given in Table.

Activity	Preceded by Initial activity	Duration (weeks)
a	-	10
b	a	9
c	a	7
d	b	6
e	b	12
f	c	6
g	c	8
h	f	8
i	d	4
j	g,h	11
k	e	5
l	i	7

Number the events by Fulkerson's rule and find the critical path. Also find the time for completing the project.

Answers: Self Assessment

1. Optimistic
2. CPM
3. dummy activity
4. Dummy activities
5. critical activity
6. float
7. Controlling
8. Three
9. activity
10. PERT
11. creating, maintaining
12. Standard normal deviate
13. Network
14. longest
15. Most likely

12.8 Further Readings

Books

Klastorin, Ted. *Project Management: Tools and Trade-offs*. Wiley.

Milosevic, Dragan Z. *Project Management Tool Box: Tools and Techniques for the Practicing Project Manager*. Wiley.

Online links

www.imamu.edu.sa

www.netmba.com

www.mindtools.com

Notes

Unit 13: Inventory Control

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Objectives

After studying this unit, you will be able to:

- Discuss the significance of inventory management
- Analyse the decision regarding right quality and right time
- Learn about the inventory control system in practice
- Describe various methods of inventory control

Introduction

Material cost shares a large portion on investment. Out of the total cost of production, material cost tunes up to 60-70%, sometimes even more than that. Hence, it calls in for a proper, timely and systematized Materials Management, Inventory Management in itself is an area of specialization. The concept of 'Material Planning' calls in for preparation of policies like: purchase policy, storage policy and usage policy. Further framing and execution of these policies consist of activities like:

Purchase Policy

1. Purchase procedure
2. Determination of quantity of raw materials
3. Centralised and decentralized purchasing

Notes

4. Vendor development programmes and vendor rating
5. Codification and classification of materials
6. Investment policy and import substitution

Storage Policy

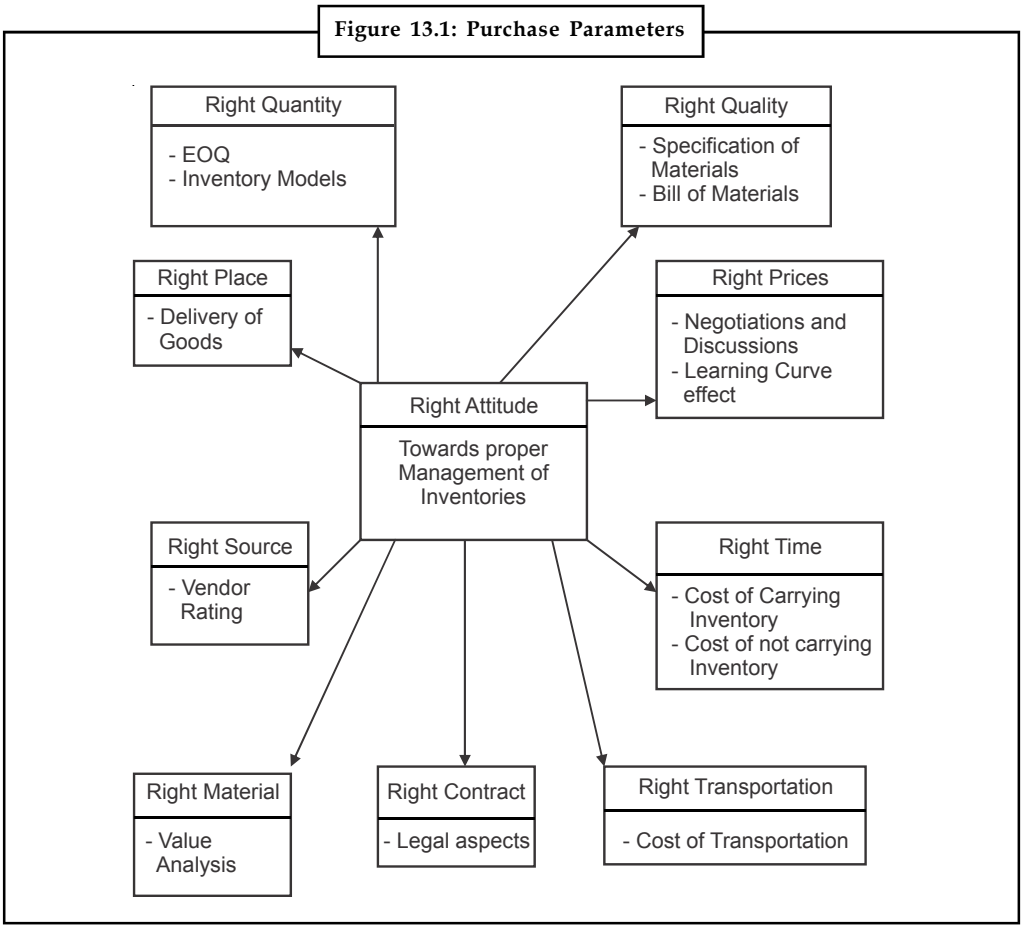
7. Centralised and decentralized storing
8. Inventory control system
9. Use of techniques like ABC Analysis, VED Analysis and FSN Analysis, etc.

Usage Policy

10. Wastage, scrap, spoilings and defectives
11. Pricing of raw materials
12. Material handling

A systematized purchase procedure is to be followed by the manufacturer for purchasing raw materials for his business. His aim should not be just to acquire the raw-material at a low price, he has to consider so many other factors like quality of raw material, quantity, timing of purchases, sources, etc. Hence, there are several purchase parameters. They are portrayed as follows:

Figure 13.1: Purchase Parameters



Notes

As it will not be accommodative and feasible to discuss all these parameters in length in this chapter, only few important of them are discussed.

Notes There are three basic questions of inventory control which every manager addresses:

1. What items should be stock?
2. When should the order be placed to replenish inventory?
3. How much should be ordered in each replenishment?

13.1 Right Quantity of Raw-material

Determination of right quantity of raw-material is a major task faced by a buyer. Say if an industry requires 1,000 units in the form of raw materials per annum, whether it is going to purchase all the 1,000 units at a time by placing a single order or whether it is going to buy once a month or once in two months or six months etc. If the cost aspect is considered here i.e., if the firm is going to order for 1,000 units at a time means, it will have to invest huge amount at a time and the storage or cost of carrying or stocking comes into picture. Even the stock will be unnecessarily piled-up and the working capital gets blocked-up irrespective of the fact that the large buying is always cushioned by quantity discount. Otherwise, if the firm is going to order say – twelve times for 1,000 units, then the ordering cost or cost of placing an order every time, the availability of required raw-material, price fluctuation, etc., come-up into picture. In other words, if the firm buys the raw-material, in large quantities, the carrying cost will be high and if it buys frequently or at frequent intervals to avoid overstocking, the ordering cost will be high. The next question is how to balance the cost at a reasonable point? The emergence of Economic Order Quantity (EOQ) concept has helped in answering the question. EOQ, as the very name says that the order quantity which is about to be placed is economical.

Two types of costs are associated with the order quantity are:

1. ***Ordering Cost:*** The cost which is incurred to place an order is ordering cost. For example, clerical expenses, expenses for scrutinizing the quotations, cost of inspection, follow-up etc. Hence, the total ordering cost is determined with the help of a mathematical formula:

$$\left\{TOC = O.C./Order \times \frac{\text{Annual Usage}}{\text{Order Size}/\text{Quantity}}\right\} \text{or} \left\{TOC = \frac{O \times U}{Q}\right\}$$

Where, T.O.C = Total Ordering Cost

O.C./Order = Ordering Cost per Order

U = Annual usage

Q = Order quantity

2. ***Carrying Cost:*** The cost incurred in stocking or storing the lot purchased is the carrying cost. For example, store-keeping cost, material handling cost, insurance, cost of risks; loss of interest on capital invested on inventory, etc. Usually, this is represented in terms of percentage on purchase price of average quantity to be held and is determined with the help of a mathematical formula:

$$\left\{T.C.C = C.C\% \times \text{Purchase Price} \times \frac{\text{Order Quantity}}{2}\right\}$$

Notes

or $$T.C.C. = \frac{C.P.Q.}{2}$$

Where, T.C.C. = Total Carrying Cost

C.C% = Carrying Cost Percentage

P.P. = Purchase Price

Q/2 = Average Quantity

Therefore, the EOQ is a point where the total carrying cost and total ordering cost are at an equilibrium point. And it is given by a mathematical formula:

$$E.O.Q. = \sqrt{\frac{2\ O.U}{C.P}}$$

Where, O = Ordering Cost per Order

U = Annual Usage

C = Carrying Cost

P = Purchase Price per Unit

A simple derivation of EOQ:

$$EOQ = CPO/2 = OU/Q$$

$$CPQ/2 \times OU/Q = CPQ2 = 2OU$$

$$Q2 = 2OU/CP$$

$$Q = \sqrt{\frac{2\ OU}{CP}}$$

There are different approaches to study the EOQ concept under two situations, viz.,

1. ***In the absence of quantity discount***
   - (i) Equational/Mathematical Approach
   - (ii) Tabular/Trial and Error Approach
   - (iii) Graphical Approach
2. ***In the presence of quantity discount***
   - (i) Cost Comparison Approach
   - (ii) Price-break Approach

Did u know? Inventory has been viewed as a necessary evil (non-earning asset) that cannot be eliminated. It is termed evil because maintaining inventory ties up money which could have been used elsewhere and also because it increases the carrying cost. However, it is considered a necessary investment to achieve workable system of production, distribution and marketing of physical goods.

Notes

13.1.1 In the Absence of Quantity Discount

Equational Approach

Under this approach, by using the mathematical equation, the EOQ is found out.

Example: The following information in an inventory problem is available:

Annual demand	2,400 units
Unit price (₹)	2.40
Ordering cost (₹)	4.00
Storage cost (₹)	20% p.a.

Find the EOQ.

Solution:

Mathematical equation for calculating

$$EOQ = \sqrt{\frac{2\,OU}{CP}}$$

Where,

$$O = ₹\,4$$

$$U = 2{,}400 \text{ units}$$

$$C = 20\% \text{ p.a.}$$

$$P = ₹\,2.40$$

$$= \sqrt{\frac{2 \times ₹\,4 \times 2{,}400 \text{ units}}{20/100 \times ₹\,2.40}}$$

$$= \sqrt{\frac{19{,}200}{0.48}}$$

$$= \sqrt{40{,}000}$$

$$EOQ = 200 \text{ units.}$$

Example: A company plans to consume 760 pieces of a particular component. Past records indicate that purchasing department spent ₹ 12,555 for placing 15,500 purchase orders. The average inventory was valued at ₹ 45,000 and the total storage cost was ₹ 7,650 which included wages, rent, taxes insurance etc., related to stores department. The company borrows capital at the rate of 10% a year.

Fill in the following, if the price of the component is ₹ 12 and the Lot size is 10.

1. Purchase price per year……..
2. Purchase expenses per year……...
3. Storage expenses per year…….
4. Capital cost per year…….
5. Total cost…….

Notes

Solution:

1. Calculation of ordering cost per order

$$= \frac{₹\ 12,555}{15,500} = ₹\ 0.81 \text{ per order}$$

2. Calculation of carrying cost

$$= \frac{₹\ 7,650}{₹\ 45,000} = \times 100 = 17\%$$

(i) Purchase Price/year

= 760 pieces × ₹ 12 = ₹ 9,120

(ii) Purchase expenses per year

= 760 pieces/10 Lot size

= 76 orders

= 76 orders × Re. 0.81/order

= ₹ 61.56

3. Storage expenses per year

$$= \frac{10}{2} \times ₹\ 12 \times \frac{17}{100} = ₹\ 10.20$$

4. Capital cost per year

$$= \frac{10}{2} \times \frac{10}{100} \times 12 = ₹\ 6.00$$

5. Total Cost = (a) + (b) + (c) + (d) = ₹ 9,197.76

Example: The annual demand for an item is 3,200 units. The unit cost is ₹ 6 and inventory carrying charges 25% p.a. If the cost of one procurement is ₹ 150, determine:

1. EOQ
2. Number of orders p.a.
3. Time between two consecutive order

Solution:

1. Calculation of EOQ

$$= \sqrt{\frac{2OU}{CP}}$$

Where, O = ₹ 150 per order

U = 3,200 units

C = 25% p.a.

P = ₹ 6/unit

Notes

Therefore $$EOQ = \sqrt{\frac{2 \times ₹\ 150 \times 3{,}200 \text{ units}}{25/100 \times ₹\ 6}}$$

$$= \sqrt{\frac{9{,}60{,}000}{15}} = \sqrt{6{,}40{,}000}$$

$$EOQ = 800 \text{ units}$$

2. Number of order per annum

$$\frac{U}{EOQ} = \frac{3200 \text{ unit}}{800 \text{ unit}}$$

$$= 4 \text{ orders p.a.}$$

3. Time between two consecutive orders

$$= \frac{12 \text{ months}}{4 \text{ orders}}$$

$$= 3 \text{ months}$$

Task The annual consumption of a particular material in a factory is 3,000 units. Interest and cost of storing a unit costing ₹ 5 per unit is 10%. Cost of placing an order is ₹ 30. Find the EOQ using.

a. Equation method.

b. Tabular method.

c. Graphical method.

Self Assessment

State true or false:

1. The cost which is incurred to place an order is called as carrying cost.
2. Carrying cost is usually represented in terms of percentage on purchase price of average quantity to be held.
3. EOQ concept can be studied under two situations: in the absence of quantity discounts and in the presence of quantity discounts.

13.1.2 In the Presence of Quantity Discounts

Cost Comparison Method

Under this method, generally the total cost of ordering two lots would be considered and the quantity discount to be availed and the lot which reveals the lowest total cost inclusive of the purchase price is said to be the EOQ.

Notes

Example: A foundry regularly uses 1,000 folds per day for 250 days in a year. Folds can be purchased in the lots of 1,000 for ₹ 10 per lot or in lots of 10,000 for ₹ 96.10 per lot. Ordering costs are ₹ 10 per order and the holding costs of items in inventory are estimated to be 20% of cost per year.

1. What is the EOQ and associated annual cost assuming that only lots of 1,000 items are available?
2. What is the EOQ and the associated annual cost assuming that only lots of 10,000 items are available?
3. Compare the costs of the two answers above and state the optimal ordering policy for these folds assuming that lots of 1,000 or 10,000 items can be ordered.

Solution:

Step 1: Calculation of EOQ and the associated annual cost when only lots of 1,000 items are there

1. Usage per day 1,000 folds for 250 days

$$= 1{,}000 \times 250 \text{ days}$$

$$= 2{,}50{,}000 \text{ folds p.a.}$$

2. Purchase price: ₹ 10 per lot of 1,000 folds

$$= \frac{₹\,10}{1{,}000} = ₹\,0.01 \text{ Per unit}$$

3. Ordering cost ₹ 10 per order
4. Carrying cost is 20% p.a.

Step 2: Calculation of EOQ and the associated annual cost when only lots of 10,000 items are there

1. Usage p.a 2,50,000 folds
2. Purchase price : ₹ 96.10 per lot of 10,000 folds

$$= ₹\,\frac{96.10}{10{,}000} = ₹\,0.00961 \text{ per unit}$$

3. Ordering cost ₹ 10 per unit
4. Carrying cost is 20% p.a.

Step 3: Calculation of EOQ by using equation

$$\text{EOQ} = \sqrt{\frac{2OU}{CP}}$$

1. When purchase price is ₹ 0.01/unit

$$= \sqrt{\frac{2 \times ₹\,10 \times 2{,}50{,}000}{\frac{20}{100} \times ₹\,0.01}}$$

$$= 51{,}004 \text{ units}$$

Notes

Step 4: Calculation of Associated Annual Cost:

Total cost = Total purchase price + Total carrying cost + Total ordering cost

= (U × PP/U) + (CPQ/2) + (OU/Q)

1. When Q = 50,000 units

$$= (2,25,000 \times ₹\,0.01) + \left(\frac{20}{100} \times ₹\,0.01 \times \frac{50,000}{2}\right) + ₹\,10 \times \frac{2,50,000}{50,000}$$

$$= 2,500 + 50 + 50 = ₹\,2,600$$

2. When Q = 51.004 units

$$= (2,50,000 \times \text{Re. } 0.00961) + \left(\frac{20}{100} \times ₹\,0.00961\right) + \left(₹\,10 \times \frac{2,50,000}{51,004}\right)$$

$$= 2,402.50 + ₹\,49 + ₹\,49$$

$$= ₹\,2,500.50$$

Hence, it is better to buy in lot of 10,000 as the foundry can save ₹ 99.50. When the price to be paid for a unit of a material is fixed, the price goes on reducing when the lot size increases. That is to say the price gets broken over different lots in order to facilitate the purchase to avail quantity discount and optimize the total cost.

Price-break Approach

Under this approach, the EOQ is decided based on the total annual cost associated with different lots and the lot with least total annual cost would be chosen. Even though this method is similar to the earlier one, the glaring difference is, under this method, the quantity discount to be availed is generally be considered for various lots. In other words, the price is broken for different lot sizes. Hence it is called Price-break Approach. In other words, when the price to be paid for an unit of a material is fixed, the price goes on reducing when the lot size increases. That is to say that the price gets broken over different lots in order to facilitate the purchaser to avail quantity discount and optimize the total cost.

Example: From the following particulars with respect to a particular item of material of a manufacturing company, calculate the best quantity to order:

Order Quantities (tonne)	Price per tonne (₹)
Less than 250	6.00
250 but less than 800	5.90
800 but less than 2,000	5.80
2,000 but less than 4,000	5.70
4,000 and above	5.60

The annual demand for the material is 4,000 tonnes : stock holding costs are 20 percent of material cost p.a. The delivery cost per order is ₹ 6.00.

Notes

Solution:

Table showing different order sizes and the associated annual costs

A. Order size (O.S) Tonnes (Q)	200	250	800	2,000	4,000
B. Number of orders (U/Q where U = 4,000)	20	16	5	2	1
C. Purchase Prize (OS x PP/tonne) (₹)	(200 × 6) 1,200	(250 × 59) 1,475	(800 × ₹ 5.8) 4,640	(2,000 × 5.7) 11,400	(4,000 × 5.6) 22,400
D. Purchase Prize (value of O.S/2) (C/2)	600	738	2,320	5,700	11,200
Ordering cost (Number of orders × O.C per order) (B) × O.C is ₹ 6	(₹) 120	(₹) 96	(₹) 30	(₹) 12	(₹) 6
Carrying cost (20% of average inventory) [20/100 × (D)]	120	148	464	1140	2.240
Total cost	240	244	494	1,152	2,246
Annual cost of Material (U × PP per tonne)	24,000	23,600	23,200	22,800	22,400
Total annual cost	24,240	23,844	23,694	23,952	24,646

The total annual cost is minimum at 800 tonnes, therefore, the best quantity to be ordered is 800 tonnes.

Example: Find the optimum ordering quantity for a product for which the price breaks are as follows:

Quantity	Unit cost (₹)
0 < 80	20/- unit
80 < 100	18/- unit
100 < 200	16/- unit
200 < above	16/- unit

The monthly demand for the product is 400 units. The storage cost is 20% of the unit cost of the product and the cost of ordering is ₹ 25 per month.

Solution:

Calculation of Economic Order Quantity

1. Order size	80	100	200	400
2. No. of order placed	5	4	2	1
3. Value of materials (Q x P.P/U)	800	900	1,600	3,200
4. Value of inventory value of material / 2	800	900	1,600	3,200
5. Ordering cost (No. of order x C.P.O) (25)	125	100	50	25
6. Carrying cost	(800 x 0.2) 160	(900 x 0.2) 180	(1,600 x 0.2) 320	(3,200 x 2) 640
Total cost	285	280	370	665
Annual cost of Material	8,000	7,200	6,400	6,400
Total annual cost	8,285	7,480	6,770	7,065

Notes

Self Assessment

Fill in the blanks:

4. When the price to be paid for an unit of a material is fixed, the price goes on …….when the lot size increases.
5. Under the ……method, generally the total cost of ordering two lots would be considered and the quantity discount to be availed and the lot which reveals the lowest total cost inclusive of the purchase price is said to be the EOQ.
6. Under …… approach, the EOQ is decided based on the total annual cost associated with different lots and the lot with least total annual cost would be chosen.

13.2 Right Time

At what exact time the purchaser has to place his orders of replenishing his stock is a question? But whether replenishment is required or not, once again depends upon proper maintenance and management of materials. Because, the production activities should not be hampered due to storage of materials in the stock and even the working capital should not be blocked-up due to excessive storage of materials/unnecessary piling-up of stock. Significant factors to be taken note of are:

1. ***Lead Time (LT)***: Lead time is the time lag between indenting and receiving materials or the time gap between placing an order and receiving the materials. For example, Mr. X is a manufacturer and is going to place an order for replenishing stock and places the order on December 1st of 2009 and he receives the material on 28th February 2010. Then the LT is 2 months.

   For the purpose of controlling the LT, it is divided into:

   (i) Internal Lead Time/Administrative Lead Time (ALT) (If the lead is time created due to the influence of the internal administration, then it is (ALT).

   (ii) Suppliers LT (SLT) (If the L.T. is created due to the influence of the outside supplier, then it is S.L.T.).
2. ***Investment Policy:*** In order to determine whether the company has sufficient money to invest on inventories.
3. ***Storage Capacity and Related Costs:*** Quantity of materials the stores can hold and the related carrying costs.
4. ***Market Conditions:*** Whether the prevailing market conditions are conductive to purchase the materials is to be verified, (i.e., Market Intelligence).
5. Nature of the material to be indented
6. Rate of consumption
7. Government policy (stringent/strict or liberal)

   The stock levels are classified into five groups as under:

   (i) Maximum stock level (Mx. S.L.)

   (ii) Minimum stock level (Mn. S.L.)

   (iii) Reorder stock level (ROL)

Notes

(iv) Danger stock level (D.L.)

(v) Average stock level (A.L.)

Maximum Stock Level (Mx. S.L.)

Maximum Stock Level represents the maximum quantity of material which a store can hold at any point of time. This level is fixed such that the stock of material should exceed over and above this level resulting in overstocking of materials. Mathematical representation of level is given by,

$$[\text{Mx. S.L} = (\text{ROL} + \text{ROQ/EOQ} - \text{Mn. C} \times \text{Mn. L.T.})]$$

Mx.L = Maximum Stock Level

ROL = Reorder Level

ROQ = Reorder Quantity

Mn.C = Minimum Consumption

Mn. L.T. = Minimum Lead Time

Minimum Level (Mn. S.L.)

It represents the minimum quantity of material which must be stored at all times. This quantity is fixed in order to avoid the sudden shortage of material and which may hamper the production and is mathematical derived as:

$$[(\text{Mn.S.L}) = (\text{ROL} - \text{NC} \times \text{NLT})]$$

NC = Normal Consumption

NLT = Normal Lead Time

Reorder Level (ROL)

ROL represents a level which is fixed between the Mx. S.L. and Mn. S.L. such that when the materials reach this level, it is a signal that the purchaser has to place fresh orders for the materials. It is represented mathematically as

$$[\text{ROL} = (\text{Mx.Con} \times \text{Mx. L.T.})]$$

Mx.Con. = Maximum Consumption

Mx.LT. = Maximum Lead Time

Danger Level (D.L.)

It represents a level which is fixed below the Mn. S.L. As and when the stocks reach this level, urgent action should be taken to replenish the stock to avoid stock-outs. When the level is reached, normal supply to the Production Department would be stopped and material would be issued only on specific instructions.

Mathematically, D.L. is;

$$\text{D.L.} = (\text{Mn.C} \times \text{Mx.L.T. for emergency purchases})$$

Notes

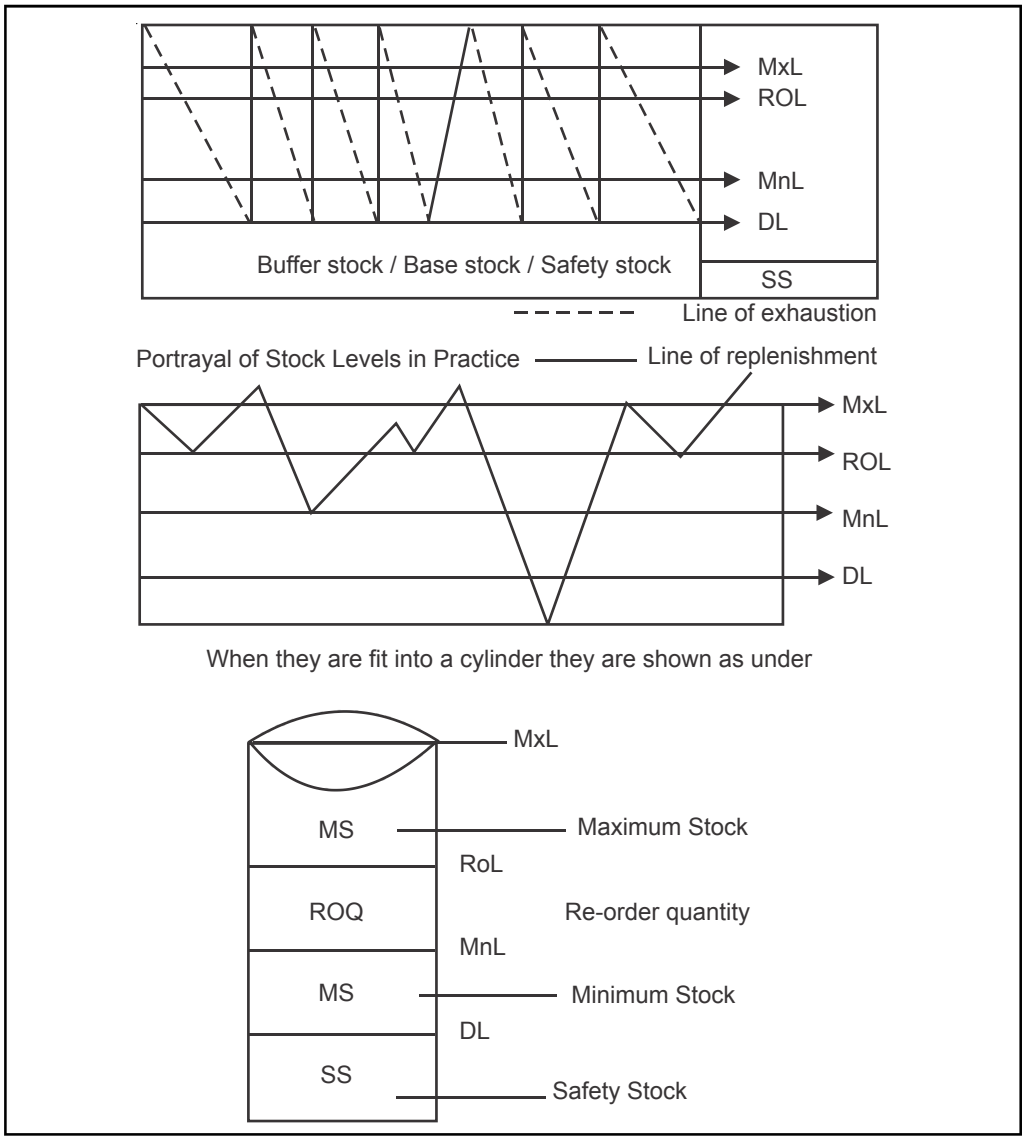
Average Level (A.L.)

This level is the average of Mn. S.L. and Mx. S.L. and is given by

$$A.L. = \frac{Mn.\ S.L. + Mx.\ S.L.}{2}$$

$$A.L. = Mn.\ S.L. + \tfrac{1}{2} \text{ of ROQ or EOQ}$$

Theoretical representation of stock levels



Example: Two components, A and B are used as follows:

Normal usage 50 units/week each

Minimum 25 units/week each

Maximum usage 75 units/week each

Notes

Reorder quantity A: 300 units

B: 500 units

Reorder period A: 4 to 6 weeks

B: 2 to 4 weeks

Calculation for each component

1. Reorder level
2. Minimum level
3. Maximum level
4. Average level

Solution:

Tabular Presentation of stock levels

Levels	A	B
(a) Re-order Level (Mx. Con × Mx. LT)	(75 × 6) = 450 units	(75 × 4 weeks) = 300 units
(b) Min. Level (ROL – NC x NLT)	(450) – (50 × 5) = 200 units	(300 – (50 × 3) = 150 units
(c) Max. Level (ROL + ROQ/EOQ – Mn.C x Mn.LT)	(450 + 300) – (25 × 4) = 650 units	(300 + 500) – (25 × 2) = 750 units
(d) Average Stock Level Mn.L + Mx.L/2 or Mn.L + ½ ROQ	$\frac{(200 + 650)}{2}$ = 425 units 200 + 1/2 (300) = 350 units	$\frac{(150 + 750)}{2}$ = 450 units 150 + ½ (500) = 400 units

Example: There are three items of materials, viz., X, Y, Z. The consumption pattern is as follows:

	X	Y	Z
Max. Consumption	1,000	1,500	1,200
Min. consumption	400	300	600
Avg. consumption	700	900	900
Re-order quantity	800	1,200	1,000
Re-order period	2-4 months	1-3 months	4-6 months

Find:

1. Reorder Level
2. Min. Level
3. Max. Level
4. Average Level

Notes

Solution:

Levels	X	Y	Z
ROL = M x C × M x LT	(1,000 × 4) = 4,000	(1,500 × 3) = 4,500	(1,200 × 6) = 7,200
Mn.L = ROL – NC x NLT	(4,000) – (700 × 3) = 1,900	(4,500) – (900 × 2) = 2,700	(7,200) – (900 × 5) = 2,700
Mx.L = ROL + ROQ – Mn.C × Mn.LT	(4,000 + 800) – (400 × 2) = 4,000	(4,500 + 1,200) – (300 × 1) = 5,400	(7,200 + 1,000) -(600 × 4) = 800
Avg. L = $\frac{M.L + M \times L}{2}$ ML + 1/2 ROQ	(1,900 + 4,000)/2 =2,950 (1,900 + $\frac{800}{2}$) =2,300	(2,700 + 5400)/2 = 4,050 (2,700 + $\frac{1,200}{2}$) = 3,300	(2,700 + 5,800)/2 =4,250 (2,700 + $\frac{1,000}{2}$) = 3,200

Task The following information is available in respect of a particular material:

Reorder Quantity 3,600 units

Max. Consumption 900 units/week

Min. Consumption 300 units/week

Normal Consumption 600 units/week

Reorder Period 3 to 5 weeks

Calculate the following:

(a) ROL (b) Mn.SL

(c) Mx. SL (d) AL

Example: Calculate the following levels:

1. Reorder Level
2. Minimum Stock Level
3. Maximum Level
4. Average Level from the following:

Normal Consumption - 60 units

Minimum Consumption - 20 units

Maximum Consumption - 80 units

Reorder Quantity:

Material X – 400 units

Material Y – 600 units

Reorder period

Material X – 4-6 weeks

Material Y – 2-4 weeks

Notes

Solution:

1. ***Reorder Level***

= Maximum Consumption × Maximum Lead Time

Material X	Material Y
(80 units × 6)	(80 units × 4)
= 480 units	= 320 units

2. ***Minimum Level***

= (Reorder Level – Normal Consumption × Normal L.T.)

(480 units – 60 × 5)	(320 units – 60 × 3)
= 180 units	= 140 units

3. ***Maximum Level***

[Reorder level + Reorder Qty. – Minimum cons. × Min. L.T.]

(480 units + 400) – (20 × 4)	= (320 + 600) – (20 × 2)
= 960 units	= 960 units

4. ***Average Level***

Maximum Level + Minimum Level

$= \frac{180 + 960}{2}$	$\frac{140 + 960}{2}$
= 570 units	= 550 units

Example: Following information available in an inventory problem are:

Annual demand	2,400 units	
Unit price (₹)		2.40
Ordering cost (₹)	4.00	
Storage cost		2% p.a.
Interest rate		10% p.a.
Lead time		½ month

Calculate EOQ, reorder level and total annual inventory cost. How much does the total inventory cost vary if the unit price is changed to ₹ 5?

Solution:

(a) Calculation of EOQ $= \sqrt{\frac{2OU}{CP}}$

$$= \sqrt{\frac{2 \times 2{,}400 \text{ units} \times ₹\ 4}{12/100 \times ₹\ 2.40}}$$

Notes

$$= \sqrt{\frac{19200}{0.288}} = 258$$

(b) Calculation of Reorder Level

$$= \text{Maximum Lead Time} \times \text{Maximum Usage}$$

$$= \frac{1}{2} \times \frac{2400}{12}$$

$$= 100 \text{ units}$$

(c) Calculation of Total Annual Inventory Cost

Cost of 2,400 units at ₹ 2.40, =	₹ 5,760.00
Ordering cost = $\frac{2,400}{258}$ = 9.3 order = or 10 orders at ₹ 4	40.00
Carrying cost of average inventory = $\frac{258}{2}$ × ₹ 0.288 =	37.15
Total cost	5837.15

(d) If the unit price is ₹ 5, then EOQ

$$= \sqrt{\frac{2 \times 2400 \times ₹\ 4}{₹\ 5 \times \frac{12}{100}}}$$

$$= \sqrt{\frac{19,200}{0.6}} = \sqrt{32,000}$$

$$= 179 \text{ units}$$

$$\text{Number of orders} = \frac{2,400}{179} = 13.4 \text{ orders or } 14 \text{ orders}$$

(e)

Total cost of inventory at the revised price	(₹)
Cost of 2,400 units at ₹ 5/unit	12,000.00
Ordering cost (14 orders ₹ 4/order	56.00
Carrying cost of average inventory = 179/2 × ₹ 5 × 12/100	53.70
Total cost…..	12,109.70

Therefore,

Difference in total inventory cost is:

(₹ 12.109.70 – 5,837.15) = ₹ 6,272.55

Example: For the following inventory problem find out:

1. How much should be ordered each time?
2. When should the order be placed?
3. What should be the inventory level (ideally) immediately before the material ordered is received?

Annual consumption (360 days) 12,000 units

Notes

Cost per unit	₹ 1
Ordering cost	₹ 12 per order
Inventory carrying cost	24%
Normal lead time	15 days
Safety stock	30 days consumption

Solution:

1. Calculation of EOQ

$$= \sqrt{\frac{2OU}{CP}}$$

$$= \sqrt{\frac{2 \times 12{,}000 \text{ units} \times ₹\ 12/\text{order}}{₹\ 1 \times 24/100}}$$

$$= 1{,}095.4 \text{ units}$$

2. Calculation of ROL.

$$= \text{Safety stock} + \text{Normal LT consumption}$$

$$= \left(\frac{12{,}000 \times 30}{360}\right) + \left(\frac{12{,}000 \times 15}{360}\right)$$

$$= 1{,}500 \text{ units}$$

3. Calculation of safety stock

$$= \frac{12{,}000 \text{ units} \times 30 \text{ days}}{360} = 1{,}000 \text{ units}$$

Self Assessment

Give one word for the following:

7. The time lag between indenting and receiving materials or the time gap between placing an order and receiving the materials.
8. The maximum quantity of material which a store can hold at any point of time.
9. Inventory level that represents that the purchaser has to place fresh orders for the materials.

13.3 Inventory Control System in Practice

There is a well know Italian saying **"take care of the pound, the penny will take care of itself"**. Of course, it is grandly true either it may be with the national scene or a company situation or human life itself. We encounter a peculiar phenomenon, out of which the vital will be few and trivial will be many. Even it is a simple common sense that to solve any important problem, rigorous efforts are to be made and vice versa. In case of inventory management too, much of the effective time is spent on managing the important materials. And the motto behind any selective inventory control is that 'equal analysis' of all the items of materials will be very costly or expensive and the concentration will also be diffused. However, the importance of any material is going to be normally decided on the basis of its cost, its critically, its availability and its rate of consumption. The credit of coining this idea goes to a German Economist' 'Pareto'. He propounded this during 1896. Hence it is popularly known as 'Pareto's Law.'

Notes

There are a number of methods of Selective Inventory Control viz., ABC Analysis (Always Better Control Analysis), VED Analysis (Vital, Essential, Desirable items, Analysis), F.S.N. Analysis (Fast Moving, Slow Moving and Non-moving items Analysis), etc.

13.3.1 ABC Analysis

Always Better Control Analysis is based on the famous 'Pareto's law. Under this, the materials in the stock/store are classified by considering three important features:

1. Vitality of the materials
2. Consumption value of the materials
3. Rate of movement.

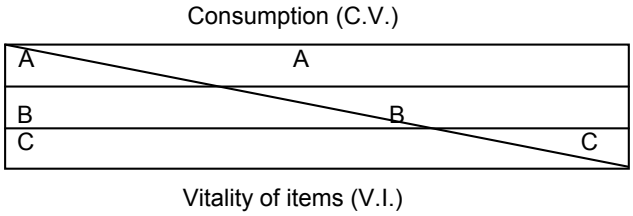
The analysis reveals that:

(i) A smaller % of items in the stores (say 10%) contribute to a larger % of value of consumption (say 70%);

(ii) A large % of items in the stores (say 70%) contribute to a small % of value of consumption (say 10%); and

(iii) Between these two extremes, the items which come, have the equal usage and consumption value.

Hence, the first category items are grouped under 'A' category, the third category items are grouped under 'C' category, and the second category items are grouped under 'B' category. The analysis can be fit into a rectangle to depict the significance of the analysis.



Let us consider a few examples of A B C Analysis and find out how the materials are classified.

Example: In Mumbai, a large scale pharmaceuticals company carries the following items on its inventory, the annual consumption and price per unit which have been shown in the following table. You have to design A, B, C analysis system for this organization for which purpose classify items in A, B and C categories on first ranking table and then prepare the table of cumulative annual usage and cumulative usage percentage. Kindly plot a suitable graph also.

Units	Annual consumption in units	Price per unit (₹)
A	5,000	10
B	650	10
C	1,800	18
D	1,200	1
E	60	8
F	6,500	6
G	2,500	5
H	4,200	1
I	50	10
J	100	7

Contd...

Notes

K	50	35
L	2,000	8
M	20	25
N	30	10
O	80	6
P	200	0.15
Q	750	0.50
R	300	4.00
S	100	7.00
T	20	10.00

Monetary limits:

Less than ₹ 1,000……...C

₹ 1,001 – 4,000 …….B

₹ 4,001 – 7,000……..A

Solution:

Calculation of Consumption Value

Name of the material	Annual consumption	Price Per unit (₹)	Consumption value (₹)	Ranking
A	5,000	10	50,000	I
B	650	10	6,500	V
C	1,800	18	35,400	III
D	1,200	1	1,200	X
E	60	8	480	XIII
F	6,500	6	39,000	II
G	2,500	5	12,500	IV
H	4,200	1	4,200	VI
I	50	10	500	XII
J	100	7	700	XI
K	100	35	3,500	VII
L	50	8	400	IVX
M	2,000	25	50,000	I
N	20	10	200	VX
O	30	6	180	VIX
P	80	0.15	12	VIIIX
Q	200	0.50	100	VIIX
R	750	4.00	3,000	VIII
S	300	7.00	2,100	IX
T	20	10	200	VX

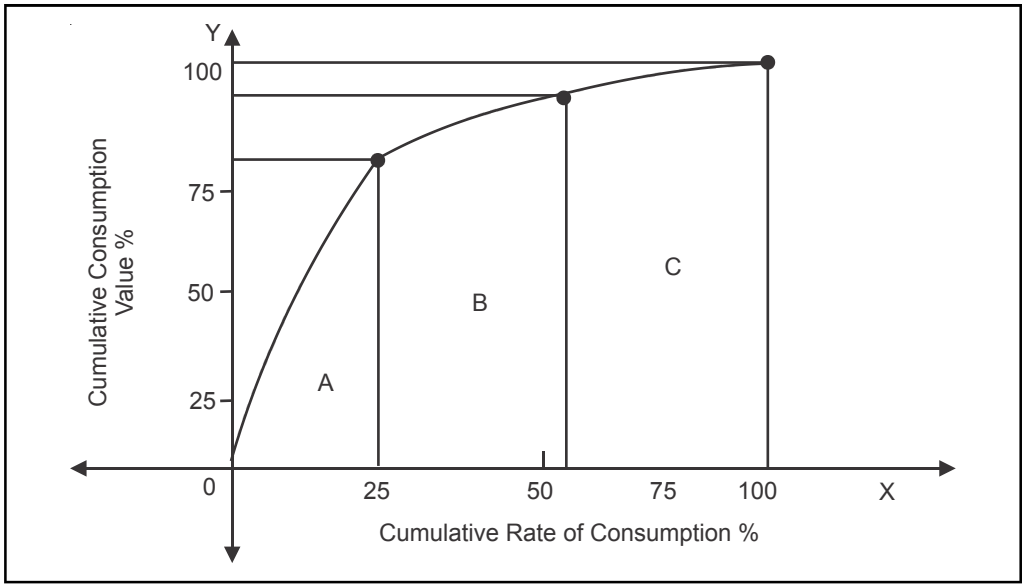
Frequency Distribution

Monetary limits	Frequency occurance		Category
Up to 1,000	~~IIII~~ ||||	9	C
Between 1,001 To 10,000	~~IIII~~ |	6	B
Above 10,001	~~IIII~~	5	A

Notes

Rank	Simple %	Cummulative	Consumption value	Cumulative c.v. %	Cumulative	Category
1–5	5/20×100 = 25%	25	1,83,900	1,83,900	88.77%	A
6–11	6/20×100 = 30%	55	20,500	2,04,400	98.67%	B
12–20	9/20×100 = 45%	100	2,772	2,07,172	100%	C

Graphical representation:



Task The following information is known about a group of materials to be classified in ABC categories:

Model No.	Annual consumption in units	Cost per unit (₹)
501	30,000	0.10
502	2,80,000	0.15
503	3,000	0.10
504	1,10,000	0.05
505	4,000	0.10
506	2,20,000	0.05
507	15,000	0.05
508	80,000	0.15
509	60,000	0.10
510	8,000	0.10

13.3.2 VED Analysis (Vital. Essential and Desirable Items Analysis)

This analysis is basically used to control spare parts keeping in view criticality of the items needed in production, the spare parts are classified in to three categories as Vital Essential and Desirable. Such spare parts, if they are not available on time to carry on production activities and virtually the activities may come to a standstill. These items are categorized as 'vital spares'. The spares, non-availability of which cannot be tolerated for more than a few hours or a day and the cost of lost production is high and at the same time 'which' are essential for the production to continue are known as 'essential spares'. The spares which are needed but their absence for a

Notes

week's time or so will not stop the production activities are known as desirable spares. Even though some spares have negligible monetary value, they may be vital for the production activities. Hence, VED Analysis helps in effective and useful classification of materials/spares.

13.3.3 FSN Analysis (Fast-moving, Slow-moving and Non-moving Items Analysis)

Comparing the turnover of different kinds of materials becomes essential in order to find out which items are fast-moving, slow-moving and non-moving, in turn to enable to avoid blocking-up of capital in such stock. Generally, a low-turnover ratio is an indicator of slow-moving items leading to obsolete materials and unnecessary piling-up of stock. On the contrary, 'a high turn over ration indicates the fast movement of stock and less investment on them. If the turnover is zero, it indicates that item has not been used during the period and there is a high chance of the material getting deteriorated. These items cause anxiety over exercising control over them because they increase the cost of material due to boosting up of carrying cost along with the material losses. The loss arising out of slow-moving stock can be reduced by reducing the quantum of material to be stored. While storing such material, the rate of consumption and the lead time should be considered.

In order to reduce the losses arising out of dormant items, only materials of absolute necessity are to be purchased. To avoid deteriorations, such materials are to be disposed off at the earliest. Thus, this classification and a periodical review of such items would help in exercising control over such materials.

Did u know? Besides ABC, VED and FSN analysis there are also a number of other techniques such as HML (high, Medium and Low), XYZ, SDE (Scarce, Difficult and Easy to obtain), GOLF (Government, Ordinary, Local and Foreign sources) , SOS (Seasonal and Off-seasonal) methods of selective inventory control.

Case Study

Ashok Distributors Ltd.

Ashok was really pleased with the 1995 operating results of Ashok Distributors Ltd. In spite of generally weak economic conditions throughout the country, Ashok Distributors Ltd., had enjoyed its most successful year. The firm was now the largest distributor of machine parts in India, with substantial sales in other countries. Its warehouses in Delhi & Mumbai had shipped over ₹ 50,000,000 worth of items in the year 2005 and would probably do even better in 2006.

As Ashok Pondered over his plan for 2006, he decided that something must be done to alleviate on the Delhi warehouse. The firm has so many items in stock that the warehouse was running out of storage space. Some items in stock that the warehouse was running out of storage space. Some items were piled in aisles and other items could not be unloaded from railway wagons since the wagons are based on daily basis, the company was incurring costly fees for the use of wagons as storage facilities.

Ashok decided to focus on a single item to see it his firm was overstocking or understocking inventory. He selected roller bearing 42Z. Ashok distributors was the major distributor for this particular item which is used as replacement part for a number, of industrial

Contd...

Notes

machines. The bearing cost only ₹ 40 and Ashok received ₹ 60 when the bearing sold. In spite of its low price, the bearing was an important item as measured by 20,000 bearings sold in 2005. These sales were fairly evenly distributed over the course of the year. Past record indicates that an average of 25 days is required for the time the order is placed until it is received at the warehouse. To guard against shortage, the firm maintains a 15 day's safety level of the part.

The firm's cost accountant provided Ashok with some additional data to consider in analyzing the situation with respect to bearing. It costs the firm approximately 10% of the bearing cost to store the bearing in the warehouse. This included insurance premium for protection against theft. Ordering the bearing including the confirmation that the bearing was naturally running low, cost the firm sum ₹ 500 per order. The 2005 average inventory of the bearing was 9,000 units and this level seemed to work fairly well.

Ashok decided to check on whether the firm really needed to store 9,000 of these bearing on an average. He knows that the average inventory should be calculated by a formula.

Average inventory = EOQ/2 + safety stock level

If this formula gave a lower inventory than 9,000 units, it might indicate that distributor was overstocked. In this case, a pruning of the inventory would allow the present warehouse to handle all the items. If the formula indicated that the firm was not overstocked, it may indicate that Ashok should begin to look for additional warehouse space. Ashok decided to find out what was happening.

Questions

1. What is the economic quantity for bearings?
2. What is the safety level?
3. What is the Reorder point?
4. Does distributors overstock or understock the bearing?
5. How many units should it stock on the average?

13.4 Summary

- Inventory Control is the art and science of maintaining the stock level of a given group of items, incurring the least total cost, consistent with other relevant targets and objectives set by the management.
- Depending upon various variables, different inventory models have been developed. Different models take different costs into account. One of the popular model developed for items of repetitive nature (dynamic), future demands for which can be projected with certainty is Economic Order Quantity (EOQ) model.
- EOQ model assumes that price of the material remains constant with time and also does not vary with order quantity. This model can be developed mathematically by differentiating total cost of inventory (ordering cost + inventory carrying cost) with respect to Quantity.
- A-B-C Classification is on the basis of consumption value of an item and does not give any importance to the criticality of the item and therefore, only A-B-C Classification is not adequate.
- VED Analysis classifies items based on their criticality. Thus the items are classified items into 3 groups called 'Vital', 'Essential' & 'Desirable'

Notes

13.5 Keywords

A-B-C Analysis :This analysis is based upon *Pareto Principle* according to which in many situations, majority of the activity (70 to 80%) is governed by very few (10 to 20) attributes.

VED Analysis: It represents classification of items based on their criticality and classifies items into 3 groups called 'Vital', 'Essential' & 'Desirable'

Vital items are those items which are very critical for the operations and do not permit any corrective time i.e. they cannot be procured off the shelf if they are not available.

Essential items are comparatively less vital and work without them cannot be managed for few days.

13.6 Review Questions

1. What are the motives for carrying inventory?
2. What is an inventory system? Give example of various types of inventory systems.
3. Describe the basic characteristics of an inventory system.
4. What are inventory models? Enumerate the various types of inventory models? Describe them briefly.
5. What are the costs associated with inventory?
6. Explain the following – Lead time, Reorder point, carrying costs.
7. Obtain an expression for the EOQ for any one inventory model, stating the assumptions made.
8. What is selective inventory control?
9. Explain ABC Analysis. What are its advantages and limitations, if any?
10. Write short notes on selective inventory control techniques.
11. XYZ Co. buys in lots of 2,000 units which is only 3 months supply. The cost per unit is ₹ 125 and the order cost is ₹ 250. The inventory cost is 20% of unit value. How much money can be saved by using Economic Order Quantity?
12. A ship building firm uses rivets at a constant rate of 20,000 numbers per year. Ordering costs are ₹ 30 per year. The rivets cost is ₹ 1.50 per number. The holding cost of rivets is estimated to be 12.5% of unit cost per year. Determine EOQ.
13. An oil engine manufacturer purchases lubricants at the rate of ₹ 4,500 per piece from a vendor. The requirements of these lubricants are 1,800 per year. What should be the order quantity per order, if the cost of placement of an order is ₹ 16.00 and inventory carrying cost per piece per year is ₹ 0.40.
14. An aircraft company uses rivets at an approximate customer rate of 2,500 kg. per year. The rivets cost at ₹ 30 per kg. and the company personnel estimate that it costs ₹ 130 to place an order and the inventory carrying cost is 10% per year. How frequently should orders for rivets be placed and what quantities should be ordered?
15. A company uses rivets at a rate of 5,000 kg. per year. Rivets cost ₹ 2 per kg. It costs ₹ 200 to place an order and the carrying cost if inventory is 10% per year. How frequently should order for rivets be placed and how much?

Notes

16. A manufacturing company uses certain part at a constant rate of 4,000 units per year. Each unit costs ₹ 2 and the company personnel estimate that it costs ₹ 50 to place an order, and that the carrying cost of inventory is 20% per year, find the optimum size of each order and the minimum yearly cost.

17. An electrical appliance manufacturer wishes to know what the economic quantity should be for a plastic impeller when following information is available:

The average daily requirement is 120 units and the company has 250 working days a year, so that the total yearly requirement is approximately 30,000 units a year. The manufacturing cost is 50 paise per part. The sum of annual rate is, interest, insurance, taxes and so forth, 20% of the unit cost and the cost of preparation is ₹ 50 per lot.

18. Find the optimum order quantity for a product for which the price breaks are as follows:

Quantity	Unit Cost (₹)
$O < Q_1 < 500$	10.00
$500 < Q_2$	9.25

The monthly demand for the product is 200 units, the cost of storage is 2% of the unit cost and the cost of ordering is ₹ 100.

19. Find the optimum order quantity for the following. Annual demand = 3,600 units, ordering cost = ₹ 50, cost of storage = 20% of the unit cost.

Price break:

Quantity	Unit Cost (₹)
$O < Q_1 < 100$	20.00
$100 < Q_2$	18.00

20. The annual demand of a product is 10,000 units. Each unit costs ₹ 100 if orders placed in quantities below 200 units but for orders of 200 or above the price are ₹ 95. The annual inventory holding cost is 10% of the value of the items and the ordering cost is ₹ 5 per order. Find the economic is size.

21. A soft drink manufacturing company buys a large number of crates every year which it uses in the warehousing of its bottled products. A local vendor has offered the following average discount schedule for crates:

Quantity	Unit Cost (₹)
Up-to 699	10
700 & above	9.25

There average yearly replacement is 2,400 crates. The carrying cost is 12% of the average inventory and ordering cost per order is ₹ 100.

22. Find the optimal order quantity for a product for which the price breaks are as follows:

Quantity	Unit Cost (₹)
$0 < Q_1 < 50$	10
$50 < Q_2 < 1000$	9
$100 < Q_3$	8

The monthly demand for the product is 200 units. The cost of storage is 25% of the unit cost and ordering cost is ₹ 20.00 per order.

Notes

Answers: Self Assessment

1.	False	2.	True
3.	True	4.	Reduced
5.	Cost Comparison method	6.	Price Break
7.	Lead Time	8.	Maximum stock Level
9.	Reorder Level		

13.7 Further Readings

Books

J.K. Sharma, *Operations Research*, *Theory and Applications,* MacMillan India Ltd.

Kanti Swarup, P.K Gupta & Manmohan, *Operations Research*, Sultan Chand Publications, New Delhi

Michael W. Carter, Camille C. Price, *Operations Research: A Practical Introduction,* CRC Press, 2001

Paul A. Jensen, Jonathan F. Bard, *Operations Research Models and Methods,* John Wiley and Sons, 2003

Richard Bronson, Govindasami Naadimuthu, *Schaum's Outline of Theory and Problems of Operations Research,* McGraw-Hill Professional; 1997

Online links

www.accountingtools.com/questions-and-answers/what-is-inventory-control.html

www.pc-education.mcmaster.ca/Tutorials.../

http://scm.ncsu.edu/scm-articles/scm-inventory-management

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Unit 14: Decision-making

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Objectives

After studying this unit, you will be able to:

- Explain the concept of decision-making under certainty, risk and uncertainty
- Determine the expected value of perfect information, expect opportunity loss and expected monetary value associated with any decision
- Discuss the concept of EVPI
- Construct decision tree for making accurate decision

Introduction

Decision-making occurs in all fields of human endeavor. A look at the headlines in the newspapers offers a glimpse of the vast range. There are economic, political, social and technical issues involved in the decision-making process. But the same decision-making process that may be observed for events worthy of inclusion on the 'Business Standards' front page underlies even the most mundane human events like eating, sleeping or going to a show.

Notes

This does not mean that the process is simple. With the advances of computing technology, intelligent machines communicate with each other. As mechanical power replaced human muscle power at the start of the industrialisation so will computing power replace brain power, which it is already started doing. A wide spectrum of decisions involving virtually all fields of human endeavor will now be made by machines. Initial emphasis in this type of decision-making has been on modeling machine decisions on human decisions. Now the emphasis is shifting to artificial intelligence process which has wide applications in the area of management as well.

All types of human decision-making are intellectual processes. This process has its roots in both the conscious as well as the subconscious mind and always involves three stages. The first one is cognition. It is the starting point for the mind that has searched for facts in the environment in order to make a decision. Cognition means discovery or recognition of data that are assembled into an information system. This assembly of recognised facts into usable information systems represents the second stage. The mind may employ convergent or divergent thinking properties in the assembly process. Finally, there is the testing stage: At this point the decision maker evaluates the cognites in terms of their relevancy to a given problem. Either a decision is made or not and any number of managerial action programs are the outcomes of this intellectual process.

There might be one or more subconscious intellectual components that affect the decision-making process negatively. The two most common flaws in decision-making are inertia and impatience. This is a paradoxical situation. Inertia is often due to a fear of change. Impatience, if regarded superficially, may appear to be somewhat of an opposite to inertia. But it has the same roots.

Some other possible major contaminants of the decision-making process either rest in the individual decision maker himself/herself or represent flaws in the general logic that is employed. In the first case mental sets are operative which are consistent as well as persistent and become especially pronounced in unstructured or ambiguous decision-making situations.

Whereas the subconscious decision-making frames of reference can be analysed in their effect only once a decision has been made, flaws in the general logic can be minimised-although not eliminated-by the appropriate design of the models and quantitative analyses that underlie the decision-making process. Moving from general to technical, one may start with tabloid thinking which represents an oversimplification of complex matter. Related to this logical flaw is the over generalisation from the specific and, unless very carefully used, the arguments form analogy, e.g., what is good for public sector is good for India. The first flaw may be called denying the antecedent, the other affirming the consequent. The first is known to the reader in its general configuration as - "if A is true, B is true and therefore if A is false, B must be false" like in "the only way to increase profits is to increase sales". A may be necessary or sufficient to make B true, but it may be quite false to argue that B could not be false without A being false. The second, affirming the consequent, is often made in forecasting studies or relationship analyses as shown later. In its general configuration it is known as - "if A is true, B is true and therefore if B is true, A is true." This statement presupposes a reversible cause and effect relationship between the two variables. However, cause and effect relationships-aside from being spurious-may be (1) necessary, (2) sufficient, (3) contingent and (4) contributory.

They are explained below:

1. Necessary implies A must exist to cause B,
2. Sufficient implies A alone could cause B but B may be caused also by C,
3. Contingent implies A causes B but only if C is present, or
4. Contributory implies A increases the probability of B's occurrence.

Notes

14.1 Decision-making under Certainty

As we already know, decision theory starts with three fundamental concepts, *viz.* actions, conditions and outcomes. While making decisions under certainty, the decision maker is fully informed, able to compute with perfect accuracy, and fully rational.

Decision under certainty means that each alternative leads to one and only one consequence, and a choice among alternatives is equivalent to a choice among consequences.

Example: There is only one possible event for the two possible actions: "Do nothing" at a future cost of $3.00 per unit for 10,000 units, or "rearrange" a facility at a future cost of $2.80 for the same number of units. A decision matrix (or payoff table) would look as in Table 14.1.

Table 14.1

Actions	State of Nature (with probability of 1.0)
Do nothing	$30,000 (10,000 units · $3.00)
Rearrange	$28,000 (10,000 units · $2.80)

There is only one State of Nature in the matrix because there is only one possible outcome for each action (with certainty). The decision is obviously to choose the action that will result in the most desirable outcome (least cost), that is to "rearrange."

Self Assessment

Name the following:

1. Different types of decision-making environment.
2. Discovery or recognition of data that are assembled into an information system.
3. Two most common flaws in decision-making
4. Decision-making environment in which each alternative leads to one and only one consequence, and a choice among alternatives is equivalent to a choice among consequences.

14.2 Decision Tree Analysis

A decision tree is a decision support tool that uses a tree-like graph or model of decisions and their possible consequences, including chance event outcomes, resource costs, and utility. A decision tree is drawn only from left to right, and has only burst nodes (splitting paths) but no sink nodes (converging paths). Therefore, if drawn manually, it can grow very big and become hard to draw fully.

A decision tree is used as a visual and analytical tool, where the expected values (or expected utility) of competing alternatives are calculated.

A decision tree consists of three types of nodes:

1. ***Decision nodes*** - commonly represented by squares
2. ***Chance nodes*** - represented by circles
3. ***End nodes*** - represented by triangles

Decision trees show the possible outcomes of different choices, taking into account probabilities, costs and returns. They enable a manager to set out the consequences of choices, ensuring that he

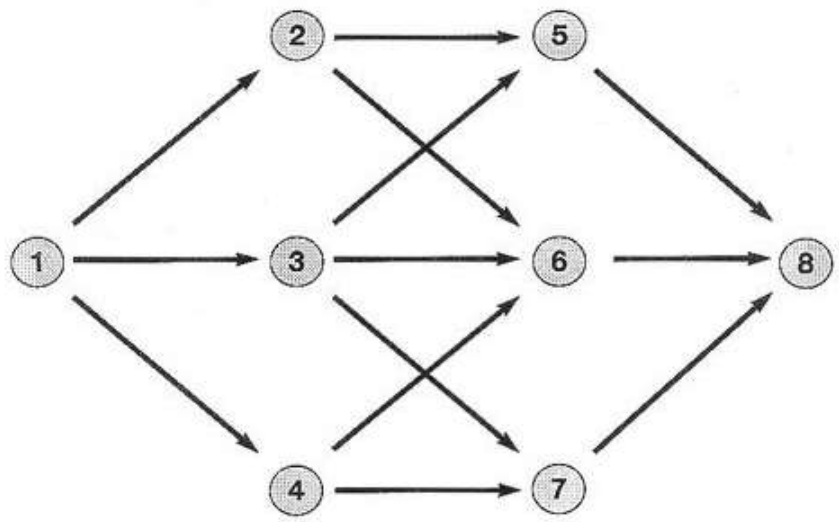
has considered all possibilities and to assess the likelihood of each different possibility and to assess the result of each possibility in terms of cost and profit.

Notes A decision tree consists of following components:

1. Nodes such as decision nodes, chance nodes and end nodes
2. Branches such as decision branch and chance branch
3. Probability estimates
4. Payoffs

An illustration of decision tree can be seen in Figure 14.1 which shows a simple decision tree for investment analysis. In the example 100,000 monetary units are to be invested and the decision is to invest either in private or public shares. The values, of course are hypothetical because the diagram represents risk and not actual returns.

Figure 14.1



You start a Decision Tree with a decision that you need to make. Draw a small square to represent this towards the left of a large piece of paper.

From this box draw out lines towards the right for each possible solution, and write that solution along the line. Keep the lines apart as far as possible so that you can expand your thoughts.

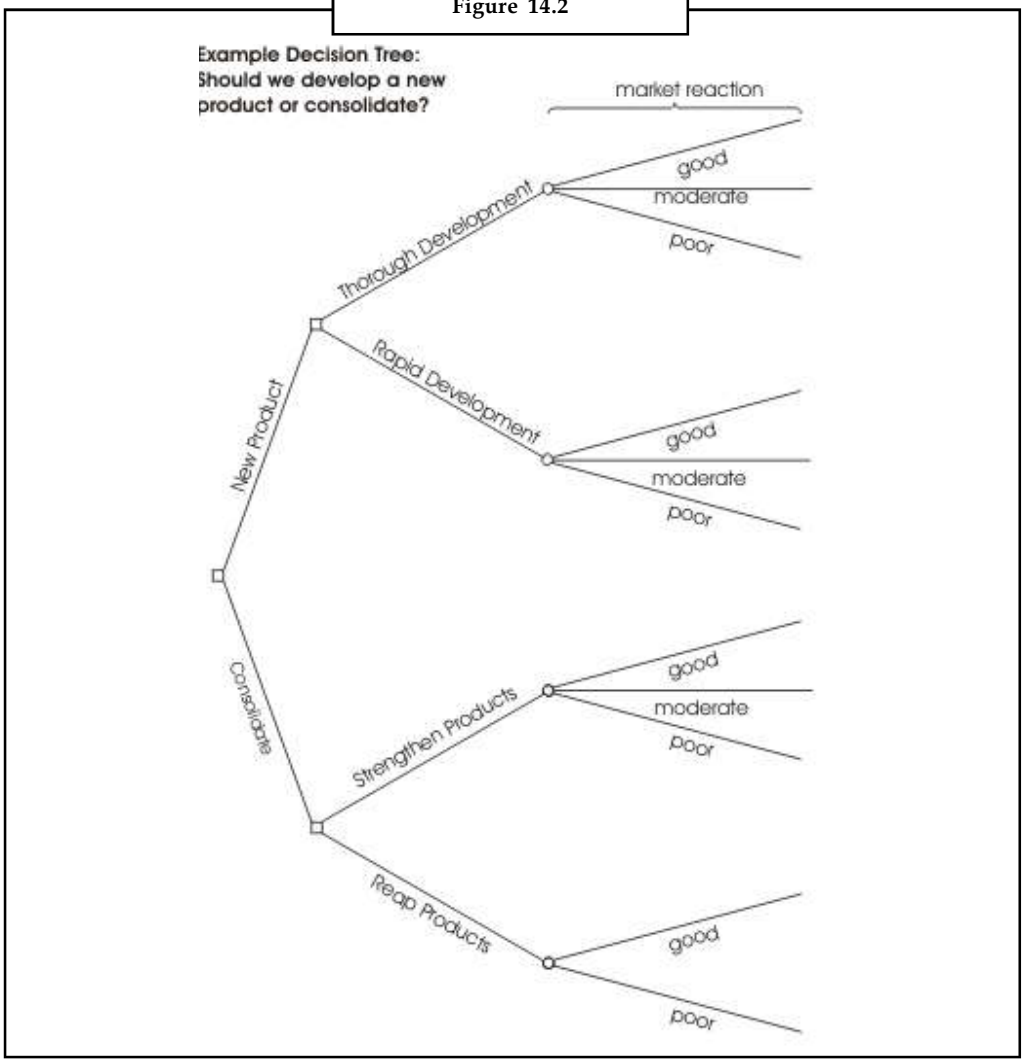
At the end of each line, consider the results. If the result of taking that decision is uncertain, draw a small circle. If the result is another decision that you need to make, draw another square. Squares represent decisions, and circles represent uncertain outcomes. Write the decision or factor above the square or circle. If you have completed the solution at the end of the line, just leave it blank.

Starting from the new decision squares on your diagram, draw out lines representing the options that you could select. From the circles draw lines representing possible outcomes. Again make a brief note on the line saying what it means. Keep on doing this until you have drawn out as many of the possible outcomes and decisions as you can see leading on from the original decisions.

Notes

An example of the sort of thing you will end up with is shown in Figure 14.2:

Figure 14.2



Once you have done this, review your tree diagram. Challenge each square and circle to see if there are any solutions or outcomes you have not considered. If there are, draw them in. If necessary, redraft your tree if parts of it are too congested or untidy. You should now have a good understanding of the range of possible outcomes of your decisions.

Evaluating Your Decision Tree

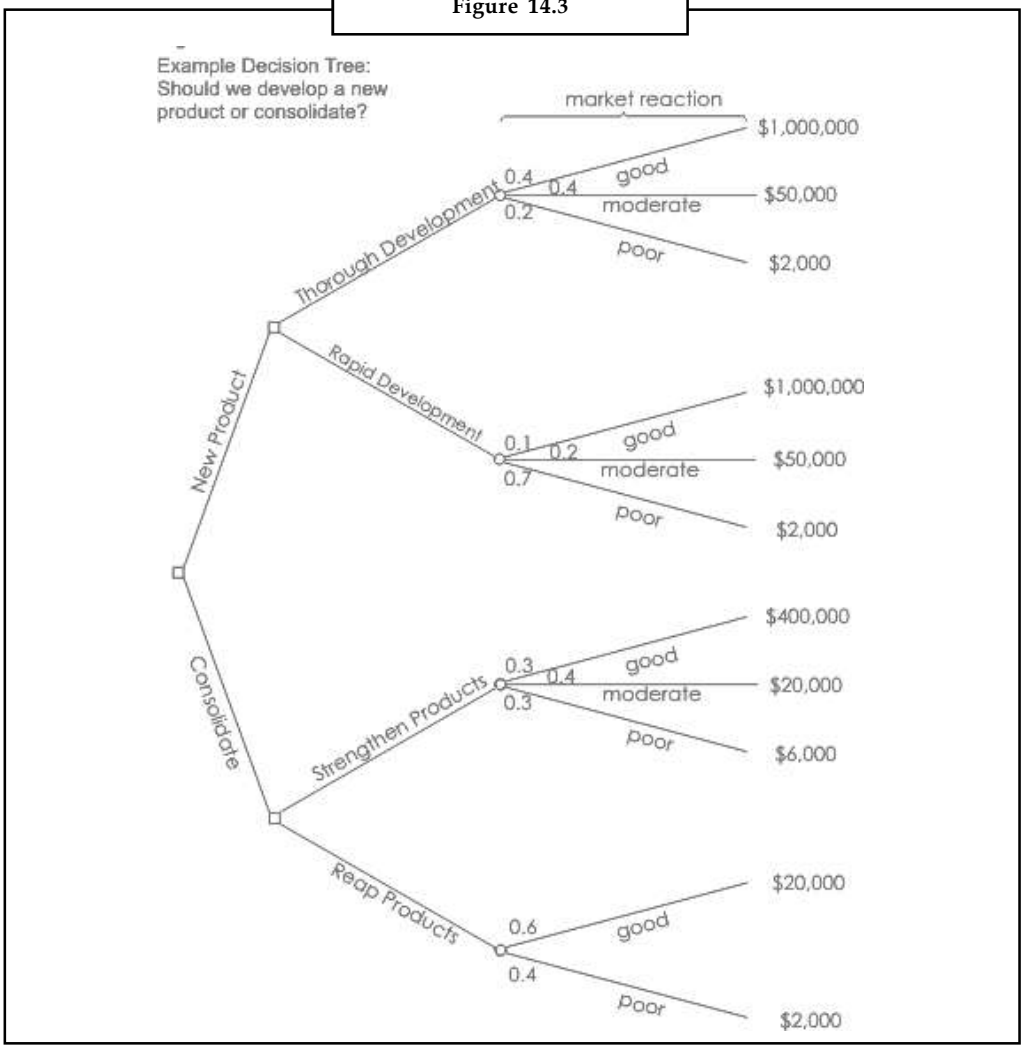
Now you are ready to evaluate the decision tree. This is where you can work out which option has the greatest worth to you. Start by assigning a cash value or score to each possible outcome. Estimate how much you think it would be worth to you if that outcome came about.

Next look at each circle (representing an uncertainty point) and estimate the probability of each outcome. If you use percentages, the total must come to 100% at each circle. If you use fractions, these must add up to 1. If you have data on past events you may be able to make rigorous estimates of the probabilities. Otherwise write down your best guess.

Notes

This will give you a tree like the one shown in Figure 14.3:

Figure 14.3



Calculating Tree Values

Once you have worked out the value of the outcomes, and have assessed the probability of the outcomes of uncertainty, it is time to start calculating the values that will help you make your decision.

Start on the right hand side of the decision tree, and work back towards the left. As you complete a set of calculations on a node (decision square or uncertainty circle), all you need to do is to record the result. You can ignore all the calculations that lead to that result from then on.

Calculating the Value of Uncertain Outcome Nodes

Where you are calculating the value of uncertain outcomes (circles on the diagram), do this by multiplying the value of the outcomes by their probability. The total for that node of the tree is the total of these values.

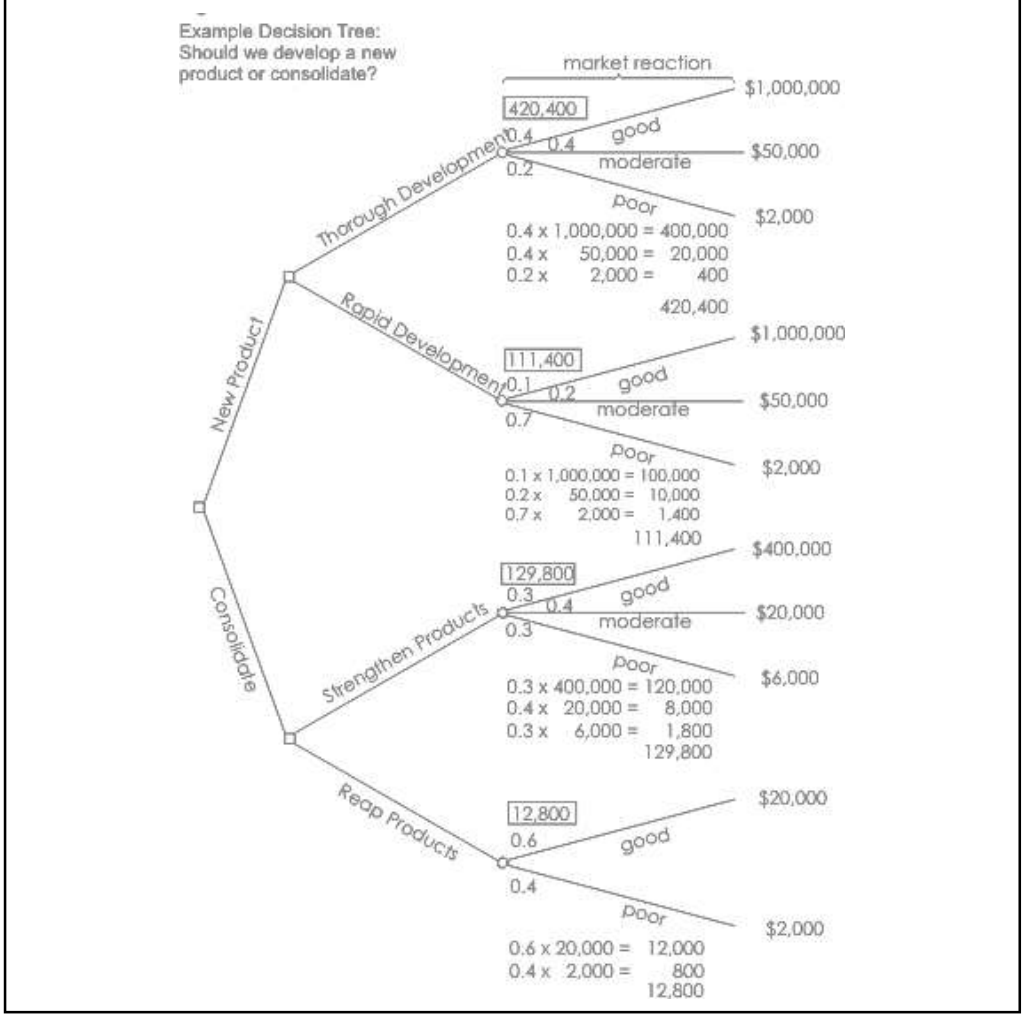
Notes

In the example in Figure 14.3, the value for 'new product, thorough development' is:

0.4 (probability good outcome) x $1,000,000 (value) =	$400,000
0.4 (probability moderate outcome) x $50,000 (value) =	$20,000
0.2 (probability poor outcome) x $2,000 (value) =	$400
	+ $420,400

Figure 14.4 shows the calculation of uncertain outcome nodes:

Figure 14.4



Note that the values calculated for each node are shown in the boxes.

Calculating the Value of Decision Nodes

When you are evaluating a decision node, write down the cost of each option along each decision line. Then subtract the cost from the outcome value that you have already calculated. This will give you a value that represents the benefit of that decision.

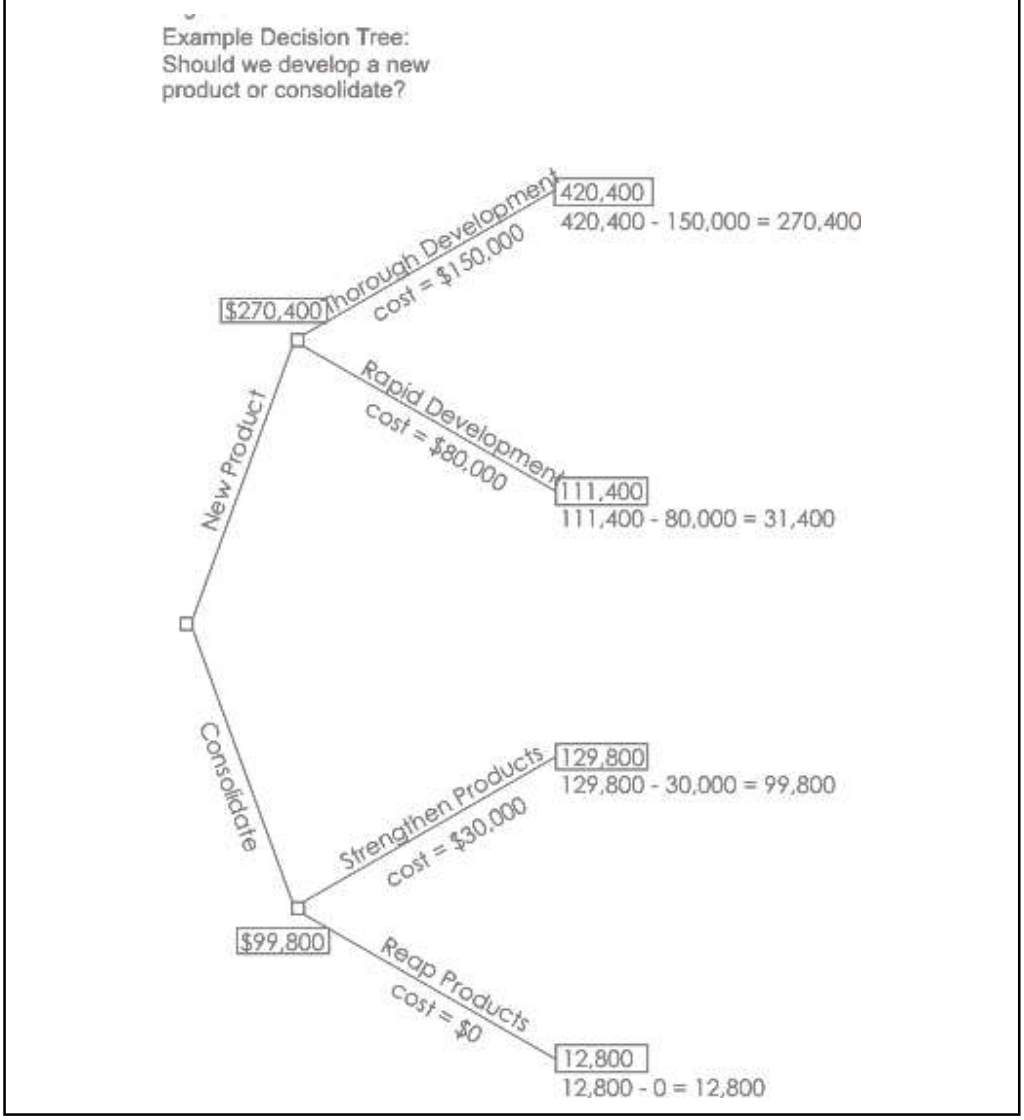
Note that amounts already spent do not count for this analysis – these are 'sunk costs' and (despite emotional counter-arguments) should not be factored into the decision.

Notes

When you have calculated these decision benefits, choose the option that has the largest benefit, and take that as the decision made. This is the value of that decision node.

Figure 14.5 shows this calculation of decision nodes in our example:

Figure 14.5



In this example, the benefit we previously calculated for 'new product, thorough development' was $420,400. We estimate the future cost of this approach as $150,000. This gives a net benefit of $270,400.

The net benefit of 'new product, rapid development' was $31,400. On this branch we therefore choose the most valuable option, 'new product, thorough development', and allocate this value to the decision node.

Notes

Result

By applying this technique we can see that the best option is to develop a new product. It is worth much more to us to take our time and get the product right, than to rush the product to market. It is better just to improve our existing products than to botch a new product, even though it costs us less.

Key Points

Decision trees provide an effective method of Decision-making because they:

- Clearly lay out the problem so that all options can be challenged.
- Allow us to analyze fully the possible consequences of a decision.
- Provide a framework to quantify the values of outcomes and the probabilities of achieving them.
- Help us to make the best decisions on the basis of existing information and best guesses.

As with all Decision-making methods, decision tree analysis should be used in conjunction with common sense – Decision trees are just one important part of your Decision-making tool kit.

Example: A private investment firm has ₹ 10 crores available in cash. It can invest the money in a bank at 10% yielding a return of ₹ 15 crore over five years (ignore compound interest).

Alternatively it can invest in mutual funds, of which there are currently two available.

If it invests in Mutual Fund A there is a 0.5 chance of it being a success yielding ₹20 crore, and a 0.5 chance of it failing leading to a loss of ₹ 5 crore. (over the five year period)

If it invests in Mutual Fund B there is a 0.6 chance of the project being a success yielding ₹ 30 crore and a 0.4 chance of it failing leading to a loss of ₹ 2 crore. (over the five year period)

Show the most feasible solution by the help of decision tree.

Solution:

Working out the likely outcomes:

Invest in bank – return = ₹ 15 cr

Expected Value of investment in Mutual Fund A

$$= E(X) = \sum_j x_j P(X = x_j)$$

= ₹ 7.5 cr

Expected Value of investment in Mutual Fund B

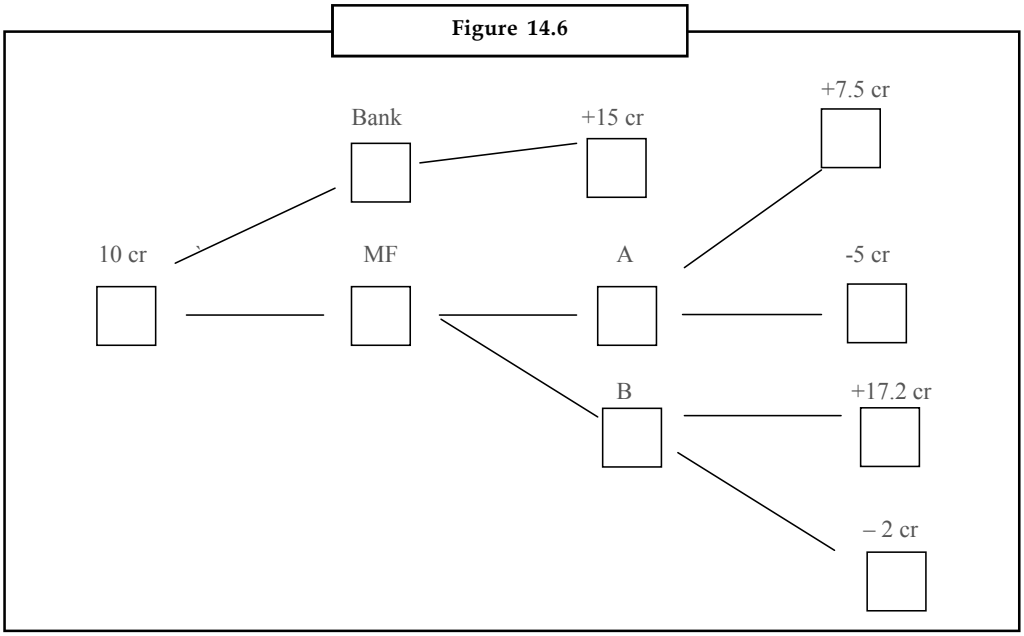
$$= E(X) = \sum_j x_j P(X = x_j)$$

= ₹ 17.2 cr

You can see that Project B yields the best result. We can illustrate this information on a decision tree as in Figure 14.2.

Notes

Figure 14.6



We set out the tree initially by working from left to right, the decision fork is to invest, or go for Mutual Fund A or B. There are then chance forks where probabilities are involved. When we have set out the tree we can prune it back by cutting off the branches which yield the worst results. This leaves us with the final expected value ₹ 17.20 cr which we put as our final decision.

Hence the most feasible decision is to make the investment in Mutual Fund B.

Self Assessment

State true or false:

5. A decision tree is used as a visual and analytical tool, where the expected values (or expected utility) of competing alternatives are calculated.

6. A decision Tree consists of 4 types of nodes.

7. End nodes are represented by triangles in a decision tree.

14.3 Decision-making under Risk

"Risk" refers to situations where the decision-maker can assign mathematical probabilities to the randomness which he is faced with.

Example: ABC Corporation wishes to introduce one of two products to the market this year. The Probabilities and Present Values (PV) of projected cash inflows are shown in Table 14.2. By the help of decision tree, suggest which product should company choose?

Notes

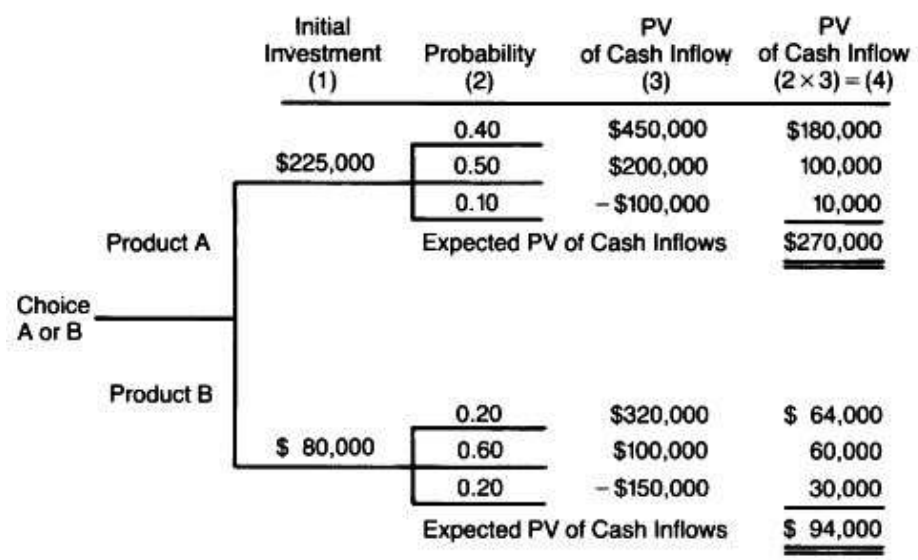
Table 14.2: Probabilities and PVs of Projected Cash Inflows

Products	Initial Investment	PV of Cash Inflows	Probabilities
A	$ 225,000		1.00
		$ 450,000	0.40
		200,000	0.50
		–100,000	0.10
B	80,000		1.00
		320,000	0.20
		100,000	0.60
		–150,000	0.20

Solution:

The decision tree analyzing the two products follows:

Decision Tree



For Product A:

Expected NPV = expected PV – I = $270,000 – $225,000 = $45,000

For Product B:

Expected NPV = $94,000 – $80,000 = $14,000

Hence based on the expected net present value, the company should choose product A over product B.

Did u know? The most widely used criterion for evaluating various courses of action under risk is the Expected Monetary Value (EMV) or Expected Value. EMV is the weighted sum of possible payoffs for each alternative.

Notes

Task Draw decision table for the following situation:

Rajesh needs to decide whether to sue his doctor not. If he does not, and he wins, he'll have the satisfaction of seeing the malicious doctor suffer humiliation and might gain enough money to pay the legal fees. If he looses, he will be bankrupted by the legal fees. But if he doesn't sue, he'll always have to wonder whether he might have gained some satisfaction by doing so.

14.4 Decision-making under Uncertainty

"Uncertainty" refers to situations when this randomness "cannot" be expressed in terms of specific mathematical probabilities.

In decision-making under pure uncertainty, the decision maker has no knowledge regarding the outcomes of any of the states of nature. In such situations, the decision-making depends merely on the decision maker's personality type and the decision maker's behaviour is purely based on his/her attitude toward the unknown. Some of these behaviours are optimistic, pessimistic, and least regret, among others.

With no knowledge regarding the likelihood (probability) of any of the events occurring, the decision maker must base his decision solely on the actual conditional payoff values, together wit his attitude or anticipation toward earning those values. Four decision criteria reflecting different attitudes will be discussed: the pessimistic, the optimistic, the equally likely, and the Savage opportunity-loss decision criteria. These criteria may lead to different decisions for the same problem; thus, it is important that the decision maker select his appropriate criterion at the outset. We shall discuss each of the uncertainty decision criteria with respect to the apple peddler problem.

14.4.1 Pessimistic Decision Criterion

The pessimistic decision criterion, or what is sometimes called the minimax or maximum criterion, assures the decision maker that he will earn no less (or pay no more) than some specified amount. It is a very conservative approach to decision-making, in that we anticipate the worst possible outcome (minimum for profit and maximum for cost) for any strategy that we might choose. The optimal strategy chosen is then the best (maximin or minimax) of the anticipated outcomes.

Steps for Decision under Pessimistic Criteria

The formal procedure for finding the pessimistic decision is as follows:

1. For each possible strategy, identify the worst payoff value. This will be the row minimum for a profit table and the row maximum for a cost table. Record this number in a new column.
2. Select the strategy with the best anticipated payoff value (maximum for profit and minimum for cost)

It should be pointed out that, this decision rule does not consider the utility values of the various outcomes nor does it allow for the superimposing of the decision maker's subjective feelings about the likelihood of the various events. Rather, this criterion is best suited for those situations, where the probabilities are not easily evaluated and the decision maker is very conservative.

When dealing with the costs, the maximum cost associated with each alternative is considered and the alternative that minimizes this maximum cost is chosen.

Notes

14.4.2 Optimistic Decision Criterion

The optimistic decision criterion, or what is sometimes called the maximax or minimin criterion, assures the decision maker that he will not miss the opportunity to achieve the greatest possible payoff or lowest possible cost. However, this decision-making behaviour usually involves the risk of a large loss. The approach is optimistic in that, we anticipate the best (maximum profit or minimum cost) possible payoff for any strategy we might choose. The optimal strategy is then the best (maximum for a profit table and minimum for a cost table) of the anticipated outcomes.

In decision problems dealing with costs, the minimum for each alternative is considered and then the alternative, which minimizes the above minimum costs, is selected. This is termed as minimin principle.

Steps for Decision under Optimistic Criteria

The formal procedure for finding the optimistic decision is as follows:

1. For each possible strategy, identify the best payoff value. Record this number in a new column.
2. Select the strategy with the best anticipated payoff value. This will be a maximum for a profit table and minimum for a cost table.

14.4.3 Savage Opportunity Loss Decision Criterion

The opportunity loss decision criterion, sometimes called the Savage minimax regret decision criterion, was proposed by the economist Savage. It assures the decision maker that the opportunities for payoff that he has missed (or lost) because of the random occurrence of a possible unfavorable event will be as small a value as is possible. The approach assumes that for each strategy-event pair, a regret (or opportunity loss) value can be computed equal to the difference between what the payoff could have been (had he chosen the optimal strategy for this event) and what it actually is for the strategy chosen and the event that has occurred.

The decision process then anticipates the worst (maximum) opportunity loss for each possible strategy and chooses as the optimal strategy the one with the minimum anticipated opportunity loss.

Steps for Decision under Savage Regret Criteria

The formal procedure for finding the opportunity-loss decision is as follows:

1. From the conditional payoff table, develop the conditional opportunity loss table as follows:
   (a) For the first possible event, identify the best possible payoff value.
   (b) For each possible strategy, subtract the actual conditional payoff value from the best value. These results are the regret or opportunity loss values for this event.
   (c) Repeat steps (a) and (b) for all possible events.
2. For each possible strategy, identify the worst or maximum regret value. Record this number as a new column.
3. Select the strategy with the smallest (minimum) anticipated opportunity loss value.

Notes

14.4.4 Equally likely Decision Criterion

The equally likely decision criterion is based on the principle of insufficient reason and is attributed to Laplace and thus, this criterion is also known as Laplace criteria. The approach assumes that the decision maker has no knowledge as to which event will occur, and thus he considers the likelihood of the different events occurring as being equal. In effect, the decision maker has assigned the same probability value to each event. This probability is equal to 1/number of events. Hence, an average or expected payoff can be computed for each possible strategy and the optimal decision will be the one with the best average payoff value.

Steps for Decision under Laplace Criteria

The formal procedure for finding the equally likely decision is as follows:

1. For each possible strategy, find the average or expected payoff by adding all the possible payoffs and dividing by the number of possible events. Record this number in a new column.
2. Select the strategy with the best average payoff value: maximum for profit and minimum for cost.

Example: A Super Bazar must decide on the level of supplies it must stock to meet the needs of its customers during Diwali days. The exact number of customers is not known, but it is expected to be in one of the four categories; 300, 350, 400 or 450 customers. Four levels of supplies are thus suggested with level j being ideal (from the viewpoint of incurred costs) if the number of customers falls in category j. Deviations from the ideal levels results in additional costs either because extra supplies are stocked needlessly or because demand cannot be satisfied. The table below provides these costs in thousands of rupees.

Table 14.3

Customer category	Supplies level			
	A_1	A_1	A	A_4
E_1	7	12	20	27
E_2	10	9	10	25
E_3	23	20	14	23
E_4	32	24	21	17

Solution:

The Laplace principle assumes that E_1, E_2, E_3, and E_4 are equally likely. Thus, the associated probabilities are given by $P(E_j) = ¼¼$ (j = 1, 2, 3, 4) and the expected costs due to deviations from the best level, for different categories of customers are:

$E(A_1) = ¼ ¼¼(7 + 10 + 23 + 32) = 18.00$

$E(A_2) = ¼ ¼¼(12 + 9 + 20 + 24) = 16.25$

$E(A_3) = ¼ ¼¼(20 + 10 + 14 + 21) = 16.25$

$E(A_4) = ¼ ¼¼(27 + 25 + 23 + 17) = 23.00$

As it is evident, that the best level of inventory is specified by the supply level A_2 or A_3.

Notes

14.4.5 Criterion of Realism (Hurwicz Criterion)

This criterion is a compromise between an optimistic and pessimistic decision criterion. To start with, a coefficient of optimism a ($0 \le a \le 1$), is selected. When a is close to one, the decision maker is optimistic about the future, and when a is close to zero the decision maker is pessimistic about the future. According to Hurwicz, select strategy which maximizes:

H = a (Maximum payoff in column) + (1 – a) (Minimum payoff in column)

Task Describe a business situation where a decision maker faces a decision-making under uncertainty and where a decision based on maximizing the expected monetary value cannot be made. How do you think the decision maker should make the required decision?

14.5 Expected Value

The expected value (or expectation value, or mathematical expectation, or mean, or first moment) of a random variable is the integral of the random variable with respect to its probability measure. For discrete random variables this is equivalent to the probability-weighted sum of the possible values, and for continuous random variables with a density function it is the probability density-weighted integral of the possible values.

Caution The term "expected value" can be misleading. It must not be confused with the "most probable value." The expected value is in general not a typical value that the random variable can take on. It is often helpful to interpret the expected value of a random variable as the long-run average value of the variable over many independent repetitions of an experiment.

The expected value may be intuitively understood by the law of large numbers: The expected value, when it exists, is almost surely the limit of the sample mean as sample size grows to infinity. The value may not be expected in the general sense - the "expected value" itself may be unlikely or even impossible (such as having 2.5 children), just like the sample mean. The expected value does not exist for all distributions, such as the Cauchy distribution.

It is possible to construct an expected value equal to the probability of an event by taking the expectation of an indicator function that is one if the event has occurred and zero otherwise. This relationship can be used to translate properties of expected values into properties of probabilities, e.g. using the law of large numbers to justify estimating probabilities by frequencies.

In general, if X is a random variable defined on a probability space (Ω, Σ, P), then the expected value of X, denoted E(X), (X), $\overline{X}$ or E(X), is defined as

$$E(X) = \int_{\Omega} X \, dP$$

where the Lebesgue integral is employed. Note that not all random variables have an expected value, since the integral may not exist. Two variables with the same probability distribution will have the same expected value, if it is defined.

If X is a discrete random variable with probability mass function p(x), then the expected value becomes $E(X) = \sum_i x_i p(x_i)$ as in the gambling example mentioned above.

Notes

If the probability distribution of X admits a probability density function f(x), then the expected value can be computed as

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

It follows directly from the discrete case definition that if X is a constant random variable, i.e. X = b for some fixed real number b, the expected value of X is also b.

The expected value of an arbitrary function of X, g(X), with respect to the probability density function f(x) is given by the inner product of f and g:

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

Using representations as Riemann-Stieltjes integral and integration by parts the formula can be restated as

$$E(g(X)) = \int_{a}^{\infty} g(x)dP(X \le x) = g(a) + \int_{a}^{\infty} g'(x)P(X > x)dx$$

$$\text{if } P(X \ge a) = 1$$

As a special case let α denote a positive real number, then

$$E(|X|^{\alpha}) = \int_{0}^{\infty} t^{\alpha-1}P(|X| > t)dt$$

In particular, for α = 1, this reduces to:

$$E(X) = \int_{0}^{\infty} \{1 - F(t)\}dt\,,$$

$$\text{if } P[X \ge 0] = 1$$

14.6 EVPI

In probabilistic situation, there in no control over the occurrence of given state of nature. However, what will happen, if decision maker had exact information about the occurrence of particular state of nature.

The Expected Profit with Perfect Information (EPPI) is the maximum attainable Expected Monetary Value (EMV) based on perfect information about the state of nature that will occur. The expected profit with perfect information may be defined as the sum of the product of best state of nature corresponding to each optimal course of action and its probability.

The Expected Value of Perfect Information (EVPI) may now be defined as the maximum amount one would be willing to pay to obtain perfect information about the state of nature that would occur. EMV* represents the maximum attainable expected monetary value given only the prior outcome probabilities, with no information as to which state of nature will actually occur. Therefore, perfect information would increase profit from EMV* up to the value of EPPI. This increased amount is termed as Expected Value of Perfect Information (EVPI),

i.e., EVPI = EPPI – EMV

Example: A wholesaler of sports goods has an opportunity to buy 5,000 pairs of skiis, that have been declared surplus by the government. The wholesaler will pay ₹ 50 per pair and can obtain ₹ 100 a pair by selling skiis to retailers. The price is well established, but the wholesaler is in doubt as to just, how many pairs he will be able to sell. Any skiis left over, he can sell to discount outlets at ₹ 20 a pair. After a careful consideration of the historical data, the wholesaler assigns probabilities to the demand as shown in Table 14.4.

Notes

Table 14.4

Retailer's Demand	Probability
1000 pairs	0.6
3000 pairs	0.3
5000 pairs	0.1

1. Compute the conditional monetary and expected monetary values.
2. Compute the expected profit with a perfect predicting device.
3. Compute the EVPI.

Solution:

We are given,

Cost per pair	=	₹ 50
Selling price per pair	=	₹ 100
Profit per pair	=	₹ 50 (if, sold)
Disposal selling price	=	₹ 20 (if, unsold)
Loss on each unsold pair	=	(50 – 20) = ₹ 30

Conditional profit values are therefore computed by:

CP = 50S when ≥ D > S

50D – 30(S – D) when ≥ D < S

where, CP = conditional profit, D = pairs demanded and S = pairs stocked.

1. The resulting conditional payoffs and corresponding expected payoffs are computed in the Table 14.5.

Table 14.5

Retailer's demand (in pairs)	Probability	Conditional payoffs (₹) (Stock per week)			Expected payoffs (₹) (Stock per week)		
		1000 pairs	3000 pairs	5000 pairs	1000 pairs	3000 pairs	5000 pairs
1000	0.6	50	-10	-70	30	-6	-42
3000	0.3	50	150	90	15	45	27
5000	0.1	50	150	250	5	15	25
				EMV	**50**	**54**	**10**

Thus, max EMV (EMV*) = 54

2. The Expected Profit with Perfect Information (EPPI) is computed below:

Table 14.6

Retailer's demand (In pairs)	Probability (i)	Conditional payoffs			Under perfect information	
		1000 pairs (ii)	3000 pairs (iii)	5000 pairs (iv)	Max payoff (v)	Expected payoff (i) 2 × (v)
1000	0.6	50	-10	-70	50	30
3000	0.3	50	150	90	150	45
5000	0.1	50	150	250	250	25
					EPPI	**100**

Notes

3. EVPI = EPPI – EMV* = 100 – 54 = 46

   Thus, EVPI = ₹ 46,000

Self Assessment

Fill in the blanks:

8. In ..... situation, there in no control over the occurrence of given state of nature.
9. The ..... value may be intuitively understood by the law of large numbers.
10. The ..... may be defined as the maximum amount one would be willing to pay to obtain perfect information about the state of nature that would occur.
11. A ..... is used as a visual and analytical tool, where the expected values (or expected utility) of competing alternatives are calculated.
12. The ..... criterion assures the decision maker that he will earn no less (or pay no more) than some specified amount.
13. Decision under certainty means that each alternative leads to .....
14. All types of human decision-making are ..... processes.

14.7 Summary

- Decision-making can be regarded as an outcome of mental processes (cognitive process) leading to the selection of a course of action among several alternatives.
- Every decision-making process produces a final choice.
- The output can be an action or an opinion of choice.
- Decision analysis includes procedures, methods, and tools for: (a) identifying, representing, and assessing the important aspects of a decision situation, for (b) prescribing the recommended course of action by applying the maximum expected utility action axiom to a well-formed representation of the decision, and (c) translating the formal representation of a decision and its corresponding recommendation into insight for the decision maker and other stakeholders.
- The decision analysis helps a decision maker to choose the most feasible one from among all the alternatives available under the uncertainty they face and the situations of risk they are threatened with.
- Given a carefully defined problem, a conceptual model is generated first.
- This is followed by the selection of the appropriate quantitative model that may lead to a solution.
- Lastly, a specific algorithm is selected.
- A decision tree is one of the operations research tools that can be used to assist management in making decisions under uncertainty.
- The decision tree provides a graphical representation of the decision-making process.
- Utility is a measure of the total worth of a particular outcome.
- It reflects the decision maker's attitude toward a collection of factors such as profit, loss and risk.

Notes

- Comparing the utility analysis for a risk taker with a risk avoider, we see that, even with the same decision problem, different attitudes toward risk can lead to different recommended decisions.

14.8 Keywords

Acquiescence: Doing everything as asked without questioning the logic.

Actions: These are the options an agent ponders. Actions are often represented by rows in a table.

Conditions: These are how things turn out independently of actions. Conditions are represented by columns.

Dishonesty: Trying to obtain someone else's decision; trying to anticipate the outcome without actually going through the intellectual stages.

Falsification: Not being able to solve the problem, illegible recording of calculations and final outcomes.

Gambling: Trying to fit decision variable configurations in the process of decision-making by trial and error; assuming that the problem cannot be solved.

Impatience: Skipping the process of analysis and reaching a conclusion without any backup reasoning.

Inertia: Skipping the study of facts because it is "not important" promising oneself to come back to it later; being "confused" because certain things may not be clear.

Outcomes: These are the states that result from actions under various conditions.

Semantics: Calling the decision-making situation totally absurd, very tough, too much time-consuming.

14.9 Review Questions

1. Consider the decision problem with the profit payoff table with four decision alternatives and three states of nature.

States of Nature

Alternative	X	Y	Z
A	4	3	3
B	5	2	5
C	5	6	2
D	6	1	4

   (a) If the decision maker knows nothing about the probabilities of the three states of nature, what is the recommended decision using the minimax rule.

   (b) Assume that the payoff table provides cost rather than profit payoffs. What is the recommended decision now?

2. Consider the decision problem with the payoff table with four decision alternatives and three states of nature.

Notes

States of Nature

Alternative	X	Y	Z
A	4	3	3
B	5	2	5
C	5	6	2
D	6	1	4

(a) Draw a decision tree for the above payoff table.

(b) If the probability of the states of nature is 0.3, 0.4 and 0.3 for the states X, Y and Z respectively, find out the optimum decision.

(c) Assume that the payoff table provides cost rather than profit payoffs. What is the recommended decision now if the probabilities given in the option (b) still holds?

3. Which action would an optimal Decision Theoretic Agent take in the following situation?

	Utility of Resulting State	Probability
Action 1	10	0.2
Action 2	10000	0.001
Action 3	5	0.799

4. After your yearly checkup, the doctor has bad news and good news. The bad news is that you tested positive for a serious disease, and that the test is 99% accurate (i.e. the probability of testing positive given that you have the disease is 0.99, as is the probability of testing negative given you don't have the disease). The good news is that this is a rare disease, striking only 1 in 10,000 people. Why is it good news that the disease is rare? What are the chances that you actually have the disease?

5. John's monthly consumption:

(a) X1 = £4000 if he does not get ill

(b) X2 = £500 if he gets ill (so he cannot work)

(c) Probability of illness 0.25

(d) Consequently, probability of no illness = 1 – 0.25 = 0.75

What will be the expected value?

Answers: Self Assessment

1. Certainty, Risk and uncertainty
2. Cognition
3. Inertia and impatience
4. Decision-making under certainty
5. True
6. False
7. True
8. probabilistic
9. expected
10. Expected Value of Perfect Information (EVPI)
11. decision tree
12. pessimistic decision
13. one and only one consequence
14. intellectual

Notes

14.10 Further Readings

Books

Clemen, Robert, *Making Hard Decisions: An Introduction to Decision Analysis*, South Western College Pub.

Hammond, J.S. and Keeney, R.L. and Raiffa, H., *Smart Choices: A Practical Guide to Making Better Decisions,* Broadway.

Raiffa, Howard, *Decision Analysis: Introductory Readings on Choices Under Uncertainty,* McGraw-Hill

Rajiv Agarwal, *Network Analysis & Operations Research*, MIT Press.

Online links

www.cnx.org

ocw.mit.edu

astor.som.jhmi.edu